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Equations différentielles stochastiques rétrogrades de type champ moyen

Présentée par

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Didicace

Je dédie ce modeste travail:

- à mes chers Parents pour tout ce qu'ils m'ont donné,
- à mon mari Lazhar et mes enfants Moncef et Manissa,
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ABSTRACT

In this thesis we study the mean field reflected backward stochastic differential equation.We first establich existence and uniqueness results for MFRBDSDE when the coefficient f is Lipshitz. Secondly , we extend our results on existence when the coefficient is continuous and linear growth. Our proof are based on approximation techniques.

RESUME

Dans cette thèse, nous étudions les équations différentielles doublement stochastiques rétrogrades réfléchies de type champs moyen.Dans un premier temps, nous établissons un résultat d'existence et d'unicité quand le coefficient est Lipchitzien Deuxièmement, nous généralisons nous résultat d'existence quand le coefficient est continue et à croissance linéaire. Nos démonstrations sont basées sur des techniques d'approximation.

Introduction

After the earlier work of Pardoux & Peng [42] (1990), the theory of Backward stochastic differential equations (BSDEs in short) has a significant headway thanks to the many applications areas. Several authors contributed in weakening the Lipschitz assumption required on the drift of the equation (see Lepeltier & San Martin [32] (1997), Kobylanski [29] (2000), Mao [40] (1995), Bahlali [2,3] (2001)). Since then, these equations have found a wide field of applications as in mathematical finance, see in particular El-Karoui, Peng, Quenez [17] (1994) or in stochastic optimal control and differential games, see El-Karoui, see Hamadene and Lepeltier [23,24] (1995). They also appear to be an effective tool for constructing G-martingales on manifolds with prescribed limits, see Darling [15] (1995), and they provide probabilistic formulae for solutions of systems of quasi-linear partial differential equations, see Pardoux, Peng [43] (1992).

Cvitanic, Karatzas [14] (1995) have introduced the notion of BSDE with two reflecting barriers. This is a generalization of the work of El-Karoui et al. [20] (1997) related to the BSDE with one reflecting barrier. Roughly speaking, in [14] (1995) the authors look for a solution for a BSDE which is forced to stay between two prescribed processes L and U ($L \leq U$). Using two methods, the first one linked to Dynkin games and Picard-type iterative procedure and the second based on the penalization of an ordinary BSDE, they show that the BSDE with two reflecting barriers has a unique solution if the coeffecient (drift) of the equation is a Lipschitz map.

A new kind of Backward stochastic differential equations was introduced by Pardoux & Peng [44] (1994),

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, d\overleftarrow{B}_s - \int_t^T Z_s dW_s.$$

with two different directions of stochastic integrals, i.e., the equation involve both a standard (forward) stochastic integral dW_t and a backward stochastic integral $d\overleftarrow{B}_t$. This equation has been in order to give a probabilistic representation for the solution of the following system of semilinear parabolic SPDE

$$\begin{aligned} u(t,x) = h(x) &+ \textstyle\int_s^T \{\mathcal{L}u(r,x) + f(r,x,u(r,x),\sigma^*\nabla u(r,x)\}\, dr \\ &+ \textstyle\int_s^T g(r,x,u(r,x),\sigma^*\nabla u(r,x))\, dB_r, \quad t \le s \le T, \end{aligned}$$

such that

$$\mathcal{L} := \frac{1}{2}\sum_{i,j}(a_{ij})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}, \qquad \text{with } (a_{ij}) := \sigma\sigma^*.$$

Pardoux & Peng proved the existence and uniqueness of a solution for BDSDEs under uniformly Lipschitz conditions. Shi et al [45] (2005) provided a comparison theorem which is very important in studying viscocity solution of SPDEs with stochastic tools.

Bahlali et al [4] (2009) proved the existence and uniqueness of a solution to the following reflected backward doubly stochastic differential equations (RBDSDEs) with one continuous barrier and uniformly Lipschitz coefficients:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + K_T - K_t + \int_t^T g(s, Y_s, Z_s)\, d\overleftarrow{B}_s - \int_t^T Z_s dW_s.$$

In a recent work of Buckdahn et al. [8,9] (2009), a notion of mean-field backward stochastic differential equation (MF-BSDEs in short) of the form

$$Y_t = \xi + \int_t^T E' f(s, \omega', \omega, Y_s(\omega), Y_s(\omega'), Z_s) ds - \int_t^T Z_s dW_s,$$

with $t \in [0, T]$, was introduced. They deepened the investigation of such mean-field BSDEs by studying them in a more general framework, with general driver. They established the existence and uniqueness of solution under uniformly Lipschitz condition. The theory of mean-field BSDE has been developed by several authors, Du et al. [16] (2012) the authors established a comparison theorem and existence in the case of linear growth and continuous condition.

Mean-field Backward doubly stochastic differential equations

$$\begin{aligned} Y_t = \xi &+ \int_t^T E' f(s, \omega', \omega, Y_s(\omega), Y_s(\omega'), Z_s) ds \\ &+ \int_t^T E' g(s, \omega', \omega, Y_s(\omega'), Z_s) dB_s - \int_t^T Z_s dW_s, \end{aligned}$$

with $t \in [0, T]$, are deduced by Ruimin Xu [47] (2012), obtained the existence and uniqueness result of the solution with uniformly Lipschitz coefficients and present the connection between McKean-Vlasov SPDEs and mean-field BDSDEs.

This thesis is composed of three chapters.

The first chapter concert the study of BSDE than the Lipschitz one in both one dimension and multidimensional case. In a first time, we give an existence and uniqueness result for BSDE with Lipschitz coefficient. In a second, we consider the reflected BSDE we give the proof of the existence of solutions to RBSDEs with Lipschitz coefficient by using the penalisation method. The last topic of the first chapter is the study of the Mean field BSDE.

In chapter 2, we present some new results in the theory of backward doubly stochastic differential equations. First, under some Lipshitz assumption on the coefficient we present an existence and uniqueness results for the BDSDE. Secondly, we present, under continuous assumptions an existence result for solution of backward doubly stochastic differential equation. Note that in this chapter we define the reflected backward doubly stochastic differential equation and we recall the results of existence and uniqueness in the Lipschitz case. The existence of a maximal and a minimal solution for RBDSDEs with continuous generators in also.

Chapter 3: Our main goal is devoted to the study of the mean field backward doubly stochastic differential equations,

$$\begin{aligned} Y_t &= \xi + \int_t^T E' f\left(s,\omega,\omega',Y_s,Y_s',Z_s,Z_s'\right) ds + \int_t^T E' g\left(s,\omega,\omega',Y_s,Y_s',Z_s,Z_s'\right) dB_s \\ &\quad + K_T - K_t - \int_t^T Z_s dW_s, \end{aligned}$$

with $t \in [0,T]$.

We prove an existence result under uniformly Lipschitz condition on the coefficients, and we will show the existence of a minimal solution under weaker condition than the Lipschitz one. Precisely, we deal with continuous and linear growth coefficients. For the proof, we adapt the method of Lepeltier and San Martin which consist on of approximating the coefficients f by a sequence of Lipschitz coefficients. After, we prove existence and uniqueness results of a solution of mean field reflected backward doubly stochastic differential equations when the coefficient is Lipischtz. Finally, we will investigate this result to prove existence of maximal and minimal solution of the mean field RBDSDE with continuous and linear growth coefficient

Chapter 1

Background on Backward Stochastic Differential Equations

The objective of this chapter is to introduce the notion of backward stochastic differential equation, and to specify terminologies used in this context. We also give some basic facts, which are widely used throughout the thesis.

1.1 Definitions

Let $(\Omega,\ \mathcal{F},\, \mathbb{P})$ be a probability space on which is defined a d-dimensional Brownian motion $W := (W_t)_{t\,\leq\,T}$. Let us denote by $\left(\mathcal{F}_t^W\right)_{t\,\leq\,T}.$ the natural filtration of W and $(\mathcal{F}_t)_{t\,\leq\,T}.$its completion with the $\mathbb{P}$-null sets of $\mathcal{F}$. We define the following spaces:

$\mathcal{P}_n$	the set of $\mathcal{F}_t$- progressively measurable, $\mathbb{R}^n$- valued processes on $\Omega \times [0,T]$
$\mathbb{L}_n^2(\mathcal{F}_t)$	$= \{\eta : \mathcal{F}_t -$ measurable random $\mathbb{R}^n -$ valued variable $s.t.\ \mathbb{E}\left[|\eta|^2\right] < \infty\}$
$\mathcal{S}_n^2(0,T)$	$= \left\{\varphi \in \mathcal{P}_n \text{ with continuous paths, s.t. } \mathbb{E}\left[\sup_{t\le T} |\varphi_t|^2\right] < \infty\right\}$
$\mathcal{H}_{n\times d}^2(0,T)$	$= \left\{Z \in \mathcal{P}_n \text{ s.t. } \mathbb{E}\left[\int_0^T |Z_s|^2\, ds\right] < \infty\right\}$

Let us now introduce the notion of multi-dimensional **BSDE**.

Definition 1 *Let $\xi_T \in L_n^2(\mathcal{F}_T)$ be a $\mathbb{R}^n-$ valued terminal condition and let f be a $\mathbb{R}^n-$ valued coefficient, $\mathcal{P}_n \otimes \mathcal{B}\left(\mathbb{R}^n \times \mathbb{R}^{n\times d}\right)-$measurable. A solution for the n-dimensional* **BSDE** *associated with parameters (f, ξ_T) is a pair of progressively measurable processes $(Y,Z) := (Y_t, Z_t)_{t\,\le\,T}$ with values in $\mathbb{R}^n \otimes \mathbb{R}^{n\times d}$ such that:*

$$\begin{cases} Y \in \mathcal{S}_n^2\ , & Z \in \mathcal{H}_{n\times d}^2 \\ Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s, & 0 \le t \le T. \end{cases} \tag{1.1}$$

The differential form of this equation is

$$-dY_t = f(t, Y_t, Z_t)\, dt - Z_t dW_t, \qquad Y_T = \xi_T\ . \tag{1.2}$$

Hereafter f is called the coefficient and ξ the terminal value of the **BSDE***.*

Under some specific assumptions on the coefficient f, the **BSDE** (1.1) has a unique solution. The standard assumptions are the following:

$$(H_1)\begin{cases} (i) \quad (f(t,0,0))_{t\leq T} \in \mathcal{H}_n^2 \\ (ii) \quad f \text{ is uniformly Lipschitz with respect to } (y,z): \\ \text{there exists a constant } C \geq 0 \text{ s.t. } \forall (y,\acute{y},z,\acute{z}) \\ |f(\omega,t,y,z) - f(\omega,t,\acute{y},\acute{z})| \leq C(|y-\acute{y}| + |z-\acute{z}|), \quad dt\otimes d\mathbb{P} \quad \text{a.e.} \end{cases}$$

1.2 Existence and Uniqueness of a Solution

In [42], Pardoux and Peng have established the existence and the uniqueness of the solution of the equation (1.1) under the uniform Lipschitz condition.

Theorem 2 (*Pardoux and Peng* [42]) *Under the standard assumptions* (H_1), *there exists a unique solution* (Y,Z) *of the* **BSDE** (1.1) *with paramaeres* (f,ξ_T).

Proof. We give a proof based on a fixed point method. Let us consider the function Φ on $\mathcal{S}^2(0,T)\times\mathcal{H}_d^2(0,T)$, mapping $(U,V)\in\mathcal{S}^2(0,T)\times\mathcal{H}_d^2(0,T)$ to $(Y,Z)=\Phi(U,V)$ defined by

$$Y_t = \xi + \int_t^T f(s,U_s,V_s)\,ds - \int_t^T Z_s dW_s. \tag{1.3}$$

More precisely, the paire (Y,Z) is constructed as follows: we consider the martingale $M_t = \mathbb{E}\left[\xi + \int_0^T f(s,U_s,V_s)\,ds \backslash \mathcal{F}_t\right]$, which is square integrable under the assumptions on $(\xi,\ f)$.

We may apply the martingale representation theorem, which gives the existence and uniqueness of $Z\in\mathcal{H}_d^2(0,T)$ such that

$$M_t = M_0 + \int_0^T Z_s dW_s. \tag{1.4}$$

We then define the process Y by

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f\left(s, U_s, V_s\right) ds \backslash \mathcal{F}_t\right] = M_t - \int_0^t f\left(s, U_s, V_s\right) ds,\ \ 0 \leq t \leq T.$$

By using the representation (1.4) of M in the previous relation, and noting that $Y_T = \xi$, we see that Y satisfies (1.3).

Observe by Doob's inequality that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_t^T Z_s dW_s\right|^2\right] \leq 4\mathbb{E}\left[\int_0^T \left|Z_s\right|^2 ds\right] < \infty.$$

Under the condition on $(\xi,\ f)$, we deduce that Y lies in $\mathcal{S}^2(0,T)$. Hence, Φ is a well defined function from $\mathcal{S}^2(0,T) \times \mathcal{H}_d^2(0,T)$ into itself. Then,we see that (Y,Z) is a solution to the **BSDE** (1.1) if and only if it is a fixed point of Φ.

Let $(U,V),\ (U',V') \in \mathcal{S}^2(0,T) \times \mathcal{H}_d^2(0,T)$ and $(Y,Z) = \Phi(U,V),\ (Y',Z') = \Phi(U',V')$. We set $\left(\bar{U},\bar{V}\right) = (U-U', V-V')$ and $\bar{f}_t = f(s,U,V) - f(s,U',V')$ Take some $\beta > 0$ to be chosen later, and apply Itô's formula to $e^{\beta s}\left|\bar{Y}_s\right|^2$ between $s=0$ and $s=T$:

$$\left|\bar{Y}_0\right|^2 = -\int_0^T e^{\beta s}\left(\beta\left|\bar{Y}_s\right|^2 - 2\bar{Y}_s.\bar{f}_s\right) ds - \int_0^T e^{\beta s}\left|\bar{Z}_s\right|^2 ds - 2\int_0^T e^{\beta s}\bar{Y}_s\bar{Z}_s dW_s. \tag{1.5}$$

Observe that

$$\mathbb{E}\left[\left(\int_0^T e^{2\beta t}\left|Y_t\right|^2\left|Z_t\right|^2 dt\right)^{\frac{1}{2}}\right] \leq \frac{e^{\beta T}}{2}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|Y_t\right|^2+\int_0^T\left|Z_t\right|^2 dt\right]<\infty,$$

which shows that the local martingale $\int_0^t e^{\beta s}\bar{Y}_s.\bar{Z}_s dW_s$ is actually a uniformly integrable martingale from the Burkholder-Davis-Gundy inequality. By taking the expectation in (1.5), we get

$$\begin{aligned}\mathbb{E}\left|\bar{Y}_0\right|^2+\mathbb{E}\left[\int_0^T e^{\beta s}\left(\beta\left|\bar{Y}_s\right|^2+\left|\bar{Z}_s\right|^2\right)ds\right] &= 2\mathbb{E}\left[\int_0^T e^{\beta s}\bar{Y}_s.\bar{f}_s ds\right]\\ &\leq 2C_f\mathbb{E}\left[\int_0^T e^{\beta s}\left|\bar{Y}_s\right|\left(\left|\bar{U}_s\right|+\left|\bar{V}_s\right|\right)ds\right]\\ &\leq 4C_f^2\mathbb{E}\left[\int_0^T e^{\beta s}\left|\bar{Y}_s\right|^2\right]+\tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta s}\left(\left|\bar{U}_s\right|^2+\left|\bar{V}_s\right|^2\right)\right].\end{aligned}$$

Now, we choose $\beta=1+4C_f^2$, and obtain

$$\mathbb{E}\left[\int_0^T e^{\beta s}\left(\left|\bar{Y}_s\right|^2+\left|\bar{Z}_s\right|^2\right)ds\right]\leq\frac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta s}\left(\left|\bar{U}_s\right|^2+\left|\bar{V}_s\right|^2\right)\right].$$

This shows that Φ is a strict contraction on the Banach space $\mathcal{S}^2(0,T)\times\mathcal{H}_d^2(0,T)$ endowed with the norme

$$\|(Y,Z)\|_\beta=\left(\mathbb{E}\left[\int_0^T e^{\beta s}\left(\left|Y_s\right|^2+\left|Z_s\right|^2\right)ds\right]\right)^{\frac{1}{2}}.$$

We conclude that Φ admits a unique fixed point, which is the solution to the **BSDE** (1.1). ■

1.3 Comparison principle

We state a very useful comparison principle for BSDEs.

Theorem 3 *Let* (ξ^1, f^1) *and* (ξ^2, f^2) *be two pairs of terminal conditions and generators satisfying conditions* (H_1) *and let* $(Y^1, Z^1), (Y^2, Z^2)$ *be the solutions to their corresponding BSDEs. Suppose that:*

- $\xi^1 \leq \xi^2$ *a.s.*
- $f^1(t, Y_t^1, Z_t^1) \leq f^2(t, Y_t^2, Z_t^2)$ $dt \otimes d\mathbb{P}$ *a.e.*
- $f^2(t, Y_t^1, Z_t^1) \in \mathcal{H}^2(0, T)$.

Then $Y_t^1 \leq Y_t^2$ *for all* $0 \leq t \leq T$, *a.s.*

Furthermore, if $Y_0^2 \leq Y_0^1$, *then* $Y_t^1 = Y_t^2$, $0 \leq t \leq T$. *In particular, if* $\mathbb{P}(\xi^1 < \xi^2) > 0$ *or* $f^1(t, \cdot, \cdot) < f^2(t, \cdot, \cdot)$ *on a set of strictlly positive measure* $dt \otimes d\mathbb{P}$, *then* $Y_0^1 < Y_0^2$.

1.4 Reflected Backward Stochastic Differential Equations

Along with this section, the dimension n is equal to 1. So we are going to deal with solutions of **BSDEs** whose components Y are forced to stay above a given barrier. Let $\xi \in \mathbb{L}_1^2(\mathcal{F}_T)$ and $f(t, \omega, y, z)$ a function which satisfies the assumption (H_1). Besides let us introduce another object, called the obstacle which is a process $S := (S_t)_{t \leq T}$, continuous, $\mathcal{P}-$measurable and satisfying:

$$iii) \ \mathbb{E}\left[\sup_{0 \leq t \leq T} \left(S_t^+\right)^2\right] < +\infty.$$

Let us now introduce the notion of reflected **BSDE** (in short RBSDE) associated with (f,ξ,S). A solution for that equation is a triple of $\mathcal{P}-$measurable processes $(Y,Z,K):=(Y_t,Z_t,K_t)_{t\leq T}$, with values in $\mathbb{R}^{1+d+1}$ such that:

$$\begin{cases} Y\in\mathcal{S}_1^2, \quad \text{and } K\in\mathcal{S}_1^2\ ; \\ (iv)\quad Z\in\mathcal{H}_d^2\ \text{, in particular } \mathbb{E}\left[\int\limits_0^T |Z_t|^2\,dt\right]<\infty; \\ (v)\ \ Y_t=\xi+\int\limits_t^T f\left(s,Y_s,Z_s\right)ds+K_T-K_t-\int\limits_t^T Z_s dw_s, \qquad \forall t\in[0,T]\ ; \\ (vi)\ \ Y_t\geq S_t\ ,\ \ 0\leq t\leq T; \\ (vii)\quad \{K_t\}\ \text{ is continuous and increasing, } K_0=0 \text{ and } \int\limits_0^T \left(Y_t-S_t\right)dK_t=0. \end{cases} \tag{1.6}$$

1.4.1 Existence and Uniqueness of a Solution

Existence

Theorem 4 [***Existence***] *The reflected* **BSDE** *associated with* (f,ξ,S) *has a unique solution*

In this section, we will give a proof of theorem (4), *based on approximation via penalisation.*

In the following, C will denote a constant whose value can vary from line to line

Proof. Existence via penalization. For $n\in\mathbb{N}$, let $(Y^n,Z^n):=(Y_t^n,Z_t^n)_{t\leq T}$ be the $\mathcal{P}-$measurable processes of $\mathcal{S}_1^2\times\mathcal{H}_d^2$ such that:

$$Y_t^n=\xi+\int\limits_t^T f\left(s,Y_s^n,Z_s^n\right)ds+n\int\limits_t^T \left(Y_s^n-S_s\right)^- ds-\int\limits_t^T Z_s^n dW_s, \qquad t\leq T, \tag{1.7}$$

where ξ and f satisfy the assumptions (H_1), we define:

$$K_t^n = n\int_0^t (Y_s^n - S_s)^- \, ds, \; 0 \; \leq t \leq T.$$

It follows from the theory of BSDE's that for each n,

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t^n|^2 < \infty\right).$$

We now establish a priori estimates, uniform in n, on the sequence (Y^n, Z^n, K^n).

$$\begin{aligned}
&\mathbb{E}\left[|Y_t^n|^2 + \int_t^T |Z_s^n|^2 \, ds\right] \\
&= \mathbb{E}\,|\xi|^2 + 2\mathbb{E}\left[\int_t^T Y_s^n f\left(s, Y_s^n, Z_s^n\right) ds\right] + 2\mathbb{E}\left[\int_t^T Y_s^n n\left(Y_s^n - S_s\right)^- ds\right] \\
&= \mathbb{E}\,|\xi|^2 + 2\mathbb{E}\left[\int_t^T Y_s^n f\left(s, Y_s^n, Z_s^n\right) ds\right] + 2\mathbb{E}\left[\int_t^T Y_s^n dK_s^n\right] \\
&\leq \mathbb{E}\,|\xi|^2 + 2\mathbb{E}\left[\int_t^T \left(f\left(s,0,0\right) + K\left|Y_s^n\right| + K\left|Z_s^n\right|\right)\left|Y_s^n\right| ds\right] + 2\mathbb{E}\left[\int_t^T S_s dK_s^n\right] \\
&\leq C\left(1 + \mathbb{E}\left[\int_t^T |Y_s^n|^2 \, ds\right]\right) + \tfrac{1}{3}\mathbb{E}\left[\int_t^T |Z_s^n|^2 \, ds\right] + \tfrac{1}{\alpha}\mathbb{E}\left[\sup_{0\leq t\leq T} \left(S_t^+\right)^2\right] \\
&+\alpha\mathbb{E}\left[\left(K_T^n - K_t^n\right)^2\right],
\end{aligned}$$

where α is a universal non-negative real constant. But, for any $t \leq T$, we have,

$$K_T^n - K_t^n = Y_t^n - \xi - \int_t^T f\left(s, Y_s^n, Z_s^n\right) ds + \int_t^T Z_s^n dW_s.$$

Hence

$$\begin{aligned}
&\mathbb{E}\left[\left(K_T^n - K_t^n\right)^2\right] \\
&\leq C\mathbb{E}\left[\xi^2 + |Y_t^n|^2 + \left(\int_t^T \left|f\left(s, Y_s^n, Z_s^n\right)\right| ds\right)^2 + \left(\int_t^T Z_s^n dW_s\right)^2\right] \\
&\leq C\mathbb{E}\left[1 + \xi^2 + |Y_t^n|^2 + \int_t^T |Y_s^n|^2 \, ds + \int_t^T |Z_s^n|^2 \, ds\right].
\end{aligned}$$

Choosing $\alpha = \left(\frac{1}{3C}\right)$ we have

$$\frac{2}{3}\mathbb{E}\left(|Y_t^n|^2\right) + \frac{1}{3}\mathbb{E}\left[\int_t^T |Z_s^n|^2\, ds\right] \le C\left(1 + \mathbb{E}\left(\int_t^T |Y_s^n|^2\, ds\right)\right).$$

It then follows Gronwall's lemma that:

$$\mathbb{E}\left(\sup_{0\le t\le T}\left(|Y_t^n|^2\right) + \int_0^T |Z_t^n|^2\, dt + (K_T^n)^2\right) \le C, \qquad n \in \mathbb{N}. \tag{1.8}$$

Note that if we define

$$f_n\left(t, y, z\right) = f\left(t, y, z\right) + n\left(y - S_t\right)^-,$$

$$f_n\left(t, y, z\right) \le f_{n+1}\left(t, y, z\right),$$

and it follows from the comparison Theorem (3) that $Y_t^n \le Y_t^{n+1}, \quad 0 \le t \le T,$ $a.s.$ Hence

$$Y_t^n \uparrow Y_t, \quad 0 \le t \le T \qquad \text{a.s.}$$

and from (1.8) and Fatou's lemma we have $\mathbb{E}\left(\sup_{0\le t\le T} Y_t^2\right) \le c.$

It then follows by dominated convergence that

$$\mathbb{E}\left[\int_0^T \left(Y_t - Y_t^n\right)^2 dt\right] \to 0 \text{ as } n \to \infty. \tag{1.9}$$

Now it follows from Itô's formula that

$$\begin{aligned}\mathbb{E}\left(\left|Y_t^n - Y_t^p\right|^2\right) + \mathbb{E}\int_t^T \left|Z_s^n - Z_s^p\right|^2 ds &= 2\mathbb{E}\int_t^T \left[f\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s^p, Z_s^p\right)\right]\left(Y_s^n - Y_s^p\right) ds \\ &+2\mathbb{E}\int_t^T \left(Y_s^n - Y_s^p\right) d\left(K_s^n - K_s^p\right) \\ &\leq 2K\mathbb{E}\int_t^T \left(\left|Y_s^n - Y_s^p\right|^2 + \left|Y_s^n - Y_s^p\right| \times \left|Z_s^n - Z_s^p\right|\right) ds \\ &+2\mathbb{E}\int_t^T \left(Y_s^n - S_s\right)^- dK_s^p + 2\mathbb{E}\int_t^T \left(Y_s^p - S_s\right)^- dK_s^n,\end{aligned}$$

from which one deduces the existence of constant C such that

$$\mathbb{E}\int_t^T \left|Z_s^n - Z_s^p\right|^2 ds \leq C\mathbb{E}\int_t^T \left|Y_s^n - Y_s^p\right|^2 ds + 4\mathbb{E}\int_t^T \left(Y_s^n - S_s\right)^- dK_s^p + 4\mathbb{E}\int_t^T \left(Y_s^p - S_s\right)^- dK_s^n, \tag{1.10}$$

let us admit for a moment the following lemma

Lemma 5

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\left(Y_t^n - S_t\right)^-\right|^2\right) \longrightarrow 0 \ as\ n \to \infty.$$

We can now conclude. Indeed, (1.8) and lemma (5) imply that

$$\mathbb{E}\int_t^T \left(Y_t^n - S_t\right)^- dK_t^p + \mathbb{E}\int_t^T \left(Y_s^p - S_s\right)^- dK_s^n \to 0 \text{ as } n,\ p \to \infty,$$

hence from(1.9) and (1.10) :

$$\mathbb{E}\int_0^T \left(\left|Y_t^n - Y_t^p\right|^2 + \left|Z_t^n - Z_t^p\right|^2\right) dt \to 0 \text{ as } n,\ p \to \infty.$$

Moreover,

$$
\begin{aligned}
\left|Y_t^n - Y_t^p\right|^2 + \int_t^T \left|Z_s^n - Z_s^p\right|^2 ds \quad &= 2\int_t^T \left[f\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s^p, Z_s^p\right)\right]\left(Y_s^n - Y_s^p\right) ds \\
&+2\int_t^T \left(Y_s^n - Y_s^p\right) d\left(K_s^n - K_s^p\right) \\
&-2\int_t^T \left(Y_s^n - Y_s^p\right)\left(Z_s^n - Z_s^p\right) dW_s,
\end{aligned}
$$

and

$$
\begin{aligned}
\sup\nolimits_{0\le t\le T}\left|Y_t^n - Y_t^p\right|^2 \quad &\le 2\int_t^T \left|f\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s^p, Z_s^p\right)\right|\left|Y_s^n - Y_s^p\right| ds \\
&+2\int_0^T \left(Y_s^n - S_s\right)^- dK_s^p + 2\int_0^T \left(Y_s^p - S_s\right)^- dK_s^n \\
&+2\sup\nolimits_{0\le t\le T}\left|\int_t^T \left(Y_s^n - Y_s^p\right)\left(Z_s^n - Z_s^p\right) dW_s\right|,
\end{aligned}
$$

and from the Burkholder-Davis-Gundy inequality,

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t^n-Y_t^p|^2\right) \leq C\mathbb{E}\int_0^T\left(|Y_t^n-Y_t^p|^2+|Z_t^n-Z_t^p|^2\right)dt$$
$$+2\mathbb{E}\int_0^T(Y_t^n-S_t)^-\,dK_t^p+2\mathbb{E}\int_0^T(Y_t^p-S_t)^-\,dK_t^n$$
$$+\tfrac{1}{2}\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t^n-Y_t^p|^2\right)+C\mathbb{E}\int_0^T|Z_t^n-Z_t^p|^2\,dt.$$

Hence $\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t^n-Y_t^p|^2\right)\to 0$, as n and $p\to\infty$, and consequently from (1.7) we have

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|K_t^n-K_t^p|^2\right)\to 0, \textit{as } n \textit{ and } p\to\infty. \tag{1.11}$$

Consequently there exists a pair $(Z,\ K)$ of progressively measurable processes with values in $\mathbb{R}^d\times\mathbb{R}$ such that

$$\mathbb{E}\left(\int_0^T|Z_t-Z_t^n|^2\,dt+\sup_{0\leq t\leq T}|K_t-K_t^n|^2\right)\to 0,\ \textit{as } n\ \to\infty,$$

and (iv) and (v) are satisfied by the triple (Y,Z,K); (vi) follows from lemma(5). It remains to check (vii).

Clearly, $\{K_t\}$ is increasing. Moreover, we have just seen that (Y^n,K^n) tends to (Y,K) uniformly in t in probability. Then the measure dK^n tends to dK weakly in probability.

$$\int_0^T(Y_t^n-S_t)\,dK_t^n\to\int_0^T(Y_t-S_t)\,dK_t,$$

in probability, $as\ n\to\infty$.

We deduce from the same argument and lemma (5) that

$$\int_0^T (Y_t - S_t)\, dK_t \geq 0.$$

On the other hand,

$$\int_0^T (Y_t^n - S_t)\, dK_t^n \leq 0, \quad n \in \mathbb{N}.$$

Hence

$$\int_0^T (Y_t - S_t)\, dK_t = 0, \quad a.s.$$

and we have proved that (Y, Z, K) solves the **RBSDE**. ■

Uniqueness

We are going to show that the **RBSDE** (1.6) has a unique solution. To begin with let us deal with the uniqueness issue.

Proposition 6 (***Uniqueness***) *The reflected* **BSDE** (1.6) *associated with* (f, ξ, S) *has at most one solution.*

Proof. Assume that (Y, Z, K) and $(\bar{Y}, \bar{Z}, \bar{K})$ are tow solutions of (1.6), and define $\Delta Y = Y - \bar{Y}$, $\Delta Z = Z - \bar{Z}$, $\Delta K = K - \bar{K}$. Then, $(\Delta Y,\ \Delta Z,\ \Delta K)$ satisfies

$$\Delta Y_t = \int_t^T \left[f(s, Y_s, Z) - f\left(s, \bar{Y}_s, \bar{Z}_s\right) \right] ds - \int_t^T \Delta Z_s dW_s + \Delta K_T - \Delta K_t, 0 \leq t \leq T.$$

By applying Itô's formula to $|\Delta Y_t|^2$, and passing to expectation, we have

$$\begin{aligned}\mathbb{E}\left|\Delta Y_t\right|^2+\mathbb{E}\int_t^T\left|\Delta Z_s\right|^2 ds &= 2\mathbb{E}\int_t^T\left[f\left(s,Y_s,Z_s\right)-f\left(s,\bar{Y}_s,\bar{Z}_s\right)\right]\left(\Delta Y_s\right)ds\\ &+2\mathbb{E}\int_t^T\Delta Y_s d\Delta K_s.\end{aligned}$$

Hence

$$\begin{aligned}\int_t^T\Delta Y_s d\Delta K_s &= \int_t^T\left(Y_s-L_s+L_s-\bar{Y}_s\right)\left(dK_s-d\bar{K}_s\right)\\ &= -\int_t^T\left(Y_s-L_s\right)d\bar{K}_s-\int_t^T\left(\bar{Y}_s-L_s\right)dK_s\leq 0.\end{aligned}$$

Then we have

$$\mathbb{E}\left[\left|\Delta Y_t\right|^2+\frac{1}{2}\int_t^T\left|\Delta Z_s\right|^2 ds\right]\leq C\mathbb{E}\left[\int_t^T\left|\Delta Y_s\right|^2 ds\right].$$

By Gronwall's lemma, we conclude that $\Delta Y=0$, $\Delta Z=0$, and so $\Delta K=0$. ■

1.4.2 Comparison Result

Let us consider now another triple $\left(f',\xi'_T,S'\right)$ and assume the reflected **BSDE** associated with this triple has a solution (Y',Z',K'). The following result allows us to compare the components Y' if we can compare the triples. Namely we have:

Theorem 7 *Assume* (H_1) *holds for the coefficient* f *and that* $f'(s,Y'_s,Z'_s)\in\mathcal{H}_1^2$. *If:*

- $\mathbb{P}-a.s.,\ \xi_T\leq\xi'_T$
- $dt\otimes d\mathbb{P}-a.e.,\ f\left(t,Y'_t,Z'_t\right)\leq f'\left(t,Y'_t,Z'_t\right)$
- $S_t\leq S'_t$.

Then $\mathbb{P}-a.e., Y\leq Y'$.

Proof. Applying Itô's formula with $\left|(Y-Y')^{+}\right|^{2}$, and taking the expectation we have:

$$\begin{aligned}
&\mathbb{E}\left|(Y_t-Y_t')^{+}\right|^{2}+\mathbb{E}\int_t^T \mathbf{1}_{[Y_s>Y_s']}\left|Z_s-Z_s'\right|^{2}ds \\
&\leq 2\mathbb{E}\int_t^T (Y_s-Y_s')^{+}\left[f(s,Y_s,Z_s)-f'(s,Y_s',Z_s')\right]ds \\
&+\mathbb{E}\int_t^T (Y_s-Y_s')^{+}(dK_s-dK_s').
\end{aligned}$$

Since on $[Y_t>Y_t']$, $Y_t>S_t'\geq S_t$ we have:

$$\int_t^T \left(Y_s-Y_s'\right)^{+}\left(dK_s-dK_s'\right)=-\int_t^T \left(Y_s-Y_s'\right)^{+}dK_s'\leq 0.$$

Assume now that the lipshitz condition in the statement applies to f. Then

$$\begin{aligned}
&\mathbb{E}\left|(Y_t-Y_t')^{+}\right|^{2}+\mathbb{E}\int_t^T \mathbf{1}_{[Y_s>Y_s']}\left|Z_s-Z_s'\right|^{2}ds \\
&\leq 2\mathbb{E}\int_t^T (Y_s-Y_s')^{+}\left[f(s,Y_s,Z_s)-f'(s,Y_s',Z_s')\right]ds \\
&\leq 2K\mathbb{E}\int_t^T (Y_s-Y_s')^{+}\left[\left|Y_s-Y_s'\right|+\left|Z_s-Z_s'\right|\right]ds \\
&\leq \mathbb{E}\int_t^T \mathbf{1}_{[Y_s>Y_s']}\left|Z_s-Z_s'\right|^{2}ds+\bar{K}\mathbb{E}\int_t^T \left|(Y_s-Y_s')^{+}\right|ds.
\end{aligned}$$

Hence

$$\mathbb{E}\left|\left(Y_t-Y_t'\right)^{+}\right|^{2}\leq \bar{K}\mathbb{E}\int_t^T \left|\left(Y_s-Y_s'\right)^{+}\right|ds,$$

and from Gronwall's lemma, $(Y_t - Y_t')^+ = 0,\ 0 \leq t \leq T$. ■

1.5 Mean field Backward stochastic differential equation

This section devoted to the study of a new type of **BSDEs**, the so called Mean-Field **BSDEs**.

1.5.1 Notations and assumptions

Let $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \mathbb{P} \otimes \mathbb{P})$ be the (non-completed) product of $(\Omega, \mathcal{F}, \mathbb{P})$ with itself. We endow this product space with the filtration

$\bar{\mathbb{F}} = \left\{\bar{\mathcal{F}}_t = \mathcal{F} \otimes \mathcal{F}_t,\ 0 \leq t \leq T\right\}$. A random variable $\xi \in L^0(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^n)$ originally defined on Ω is extended canonicaly to $\bar{\Omega} : \xi'(\omega', \omega) = \xi(\omega')$,

$(\omega', \omega) \in \bar{\Omega} = \Omega \times \Omega$. For any $\theta \in L^1\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right)$ the variable $\theta(\cdot, \omega) : \Omega \to \mathbb{R}$ belongs to

$$\mathbb{E}'[\theta(\cdot, \omega)] = \int_{\Omega} \theta\left(\omega', \omega\right) \mathbb{P}\left(d\omega'\right).$$

Notice that $\mathbb{E}'[\theta] = \mathbb{E}'[\theta(\cdot, \omega)] \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and,

$$\bar{\mathbb{E}}[\theta] = \int_{\bar{\Omega}} \theta d\ \bar{\mathbb{P}} = \int_{\Omega} \mathbb{E}'[\theta(\cdot, \omega)]\, \mathbb{P}(d\omega) = \mathbb{E}\left[\mathbb{E}'[\theta]\right].$$

The driver of our mean-fieled **BSDE** is a function $f = f(\omega', \omega, t, y', z', y, z)$: $\bar{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ wich is $\bar{\mathbb{F}}-$progressively measurable, for all (y', z', y, z), and satisfies the following assumptions:

$(A1)$ There exists a constant $C \geq 0$ such that, $\bar{\mathbb{P}} - a.s.$, for all $t \in [0, T]$, $y_1, y_2, y_1', y_2' \in \mathbb{R},\ z_1, z_2, z_1', z_2' \in \mathbb{R}^d$,

$$|f(t,y_1',z_1',y_1,z_1)-f(t,y_2',z_2',y_2,z_2)|\leq C(|y_1'-y_2'|+|z_1'-z_2'|+|y_1-y_2|+|z_1-z_2|).$$

$(A2)$ $f(\cdot,0,0,0,0)\in\mathcal{H}^2_{\mathbb{F}}(0,T,\mathbb{R})$.

Remark 8 *Let $\beta:\Omega\times[0,T]\to\mathbb{R}$, $\gamma:\Omega\times[0,T]\to\mathbb{R}^d$ be two squar integrable, jointly measurable processes. Then, for our driver, we can define, for all $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, dt $\mathbb{P}(d\omega)-a.e.$,*

$$\begin{aligned} f^{\beta,\gamma}(\omega,t,y,z) &= \mathbb{E}'\left[f\left(\cdot,\omega,t,\beta_t',\gamma_t',y,z\right)\right] \\ &= \int_\Omega f(\omega',\omega,t,\beta_t(\omega'),\gamma_t(\omega'),y,z)\,\mathbb{P}(d\omega'). \end{aligned}$$

Indeed, we remark that, for all (y,z), due to our assumptions on the driver f, $f\left(\cdot,t,\beta_t',\gamma_t',y,z\right)\in\mathcal{H}^2_{\mathbb{F}}(0,T,\mathbb{R})$, and thus $f^{\beta,\gamma}(\cdot,\cdot,y,z)\in\mathcal{H}^2_{\mathbb{F}}(0,T,\mathbb{R})$. Moreover, with the constant C of assumption $(A1)$, for all $(y_1,z_1),(y_2,z_2)\in\mathbb{R}\times\mathbb{R}^d$, dt $\mathbb{P}(d\omega)-a.e.$,

$$\left|f^{\beta,\gamma}(\omega,t,y_1,z_1)-f^{\beta,\gamma}(\omega,t,y_2,z_2)\right|\leq C(|y_1-y_2|+|z_1-z_2|).$$

Consequently, there is an $\mathbb{F}-$ progressively measurable version of $f^{\beta,\gamma}(\cdot,\cdot,y,z)$, $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, such that $f^{\beta,\gamma}(\omega,t,\cdot,\cdot)$ is dt $\mathbb{P}(d\omega)-a.e.$, defined and Lipschitz in (y,z) its Lipschitz constant is that introduced in $(A1)$.

We now can state the main result of this section.

1.5.2 Existence and Uniqueness of a Solution

Theorem 9 *Under the assumptions $(A1)$ and $(A2)$, for any random variable $\xi\in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$, the mean-field* **BSDE**

$$Y_t = \xi + \int_t^T \mathbb{E}' \left[f\left(s, Y'_s, Z'_s, Y_s, Z_s\right)\right] ds - \int_t^T Z_s dW_s, \; 0 \le t \le T, \tag{1.12}$$

has a unique adapted solution

$$(Y_t, Z_t)_{t\in[0,T]} \in \mathcal{S}^2_{\mathbb{F}}(0, T, \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}\left(0, T, \mathbb{R}^d\right).$$

Remark 10 *We emphasize that, due to our notations, the driving coefficient of* (1.12) *has to be interpreted as follows*

$$\begin{aligned}\mathbb{E}'\left[f\left(s,, Y'_s, Z'_s, Y_s, Z_s\right)\right](\omega) &= \mathbb{E}'\left[f\left(s, Y'_s, Z'_s, Y_s(\omega), Z_s(\omega)\right)\right] \\ &= \int_\Omega f\left(\omega', \omega, s, Y_s(\omega'), Z_s(\omega'), Y_s(\omega), Z_s(\omega)\right) \mathbb{P}(d\omega').\end{aligned}$$

Proof. We first introduce a norm on the space $\mathcal{H}^2_{\mathbb{F}}\left(0, T, \mathbb{R}\times\mathbb{R}^d\right)$ which is equivalent to the canonical norm:

$$\|\upsilon(\cdot)\|_\beta = \left\{\mathbb{E}\int_0^T |\upsilon_s|^2 e^{\beta s} ds\right\}^{\frac{1}{2}}, \quad \beta > 0.$$

The parameter β will be specifed later.

Step 01: for any $(y, z) \in \mathcal{H}^2_{\mathbb{F}}\left(0, T; \mathbb{R}\times\mathbb{R}^d\right)$ there exists a unique solution $(Y, Z) \in \mathcal{S}^2_{\mathbb{F}}(0, T; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}\left(0, T; \mathbb{R}^d\right)$ to the following **BSDE** :

$$Y_t = \xi + \int_t^T \mathbb{E}' \left[f\left(s, y'_s, z'_s, Y_s, Z_s\right)\right] ds - \int_t^T Z_s dW_s, \; 0 \le t \le T. \tag{1.13}$$

Indeed, we define $g^{(y,z)}(s, \mu, \nu) = \mathbb{E}'\left[f\left(s, y'_s, z'_s, \mu, \nu\right)\right]$. Then, due to Remark (8), $f^{(y,z)}(s, \mu, \nu)$ satisfies (H_1), and from Theorem (2) we know there exists a unique solution $(Y, Z) \in \mathcal{S}^2_{\mathbb{F}}(0, T; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}\left(0, T; \mathbb{R}^d\right)$ to the **BSDE** (1.13).

Step 02: the result of Step 1 allows to introduce the mapping $(Y\cdot, Z\cdot) = I[(y\cdot, z\cdot)] :$ $\mathcal{H}_{\mathbb{F}}^2\left(0,T;\mathbb{R}\times\mathbb{R}^d\right) \to \mathcal{H}_{\mathbb{F}}^2\left(0,T;\mathbb{R}\times\mathbb{R}^d\right)$ by the equation

$$Y_t = \xi + \int_t^T \mathbb{E}'\left[f\left(s, y_s', z_s', Y_s, Z_s\right)\right] ds - \int_t^T Z_s dW_s,\ 0 \le t \le T. \tag{1.14}$$

For any $\left(y^1, z^1\right), \left(y^2, z^2\right) \in \mathcal{H}_{\mathbb{F}}^2\left(0,T;\mathbb{R}\times\mathbb{R}^d\right)$ we put $\left(Y^1, Z^1\right) = I\left[\left(y^1, z^1\right)\right]$, $\left(Y^2, Z^2\right) = I\left[\left(y^2, z^2\right)\right]$, $(\hat{y}, \hat{z}) = \left(y^1 - y^2, z^1 - z^2\right)$ and $\left(\hat{Y}, \hat{Z}\right) = \left(Y^1 - Y^2,\ Z^1 - Z^2\right)$. Then, by applying Itô's formula to $e^{\beta s}\left|\hat{Y}_s\right|^2$ and by using that $Y^1, Y^2 \in \mathcal{S}_{\mathbb{F}}^2(0,T;\mathbb{R})$ we get

$$\begin{aligned}
&\left|\hat{Y}_t\right|^2 + \mathbb{E}\left[\int_t^T e^{\beta(r-t)}\beta\left|\hat{Y}_r\right|^2 dr/\mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T e^{\beta(r-t)}\left|\hat{Z}_r\right|^2 dr/\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^T e^{\beta(r-t)} 2\hat{Y}_r\left(g^{\left(y^1,z^1\right)}\left(r, Y_r^1, Z_r^1\right) - g^{\left(y^2,z^2\right)}\left(r, Y_r^2, Z_r^2\right)\right) dr/\mathcal{F}_t\right],\ t \in [0,T].
\end{aligned}$$

From assumption $(A1)$ we obtain

$$\begin{aligned}
&\left(\tfrac{\beta}{2} - 2C - 2C^2\right)\mathbb{E}\left[\int_0^T e^{\beta r}\left|\hat{Y}_r\right|^2 dr\right] + \tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta r}\left|\hat{Z}_r\right|^2 dr\right] \\
&\le \tfrac{4C^2}{\beta}\left\{\mathbb{E}\left[\int_0^T e^{\beta r}\left|\hat{y}_r\right|^2 dr\right] + \mathbb{E}\left[\int_0^T e^{\beta r}\left|\hat{z}_r\right|^2 dr\right]\right\}.
\end{aligned}$$

Thus, taking $\beta = 16C^2 + 4C + 1$ we get

$$\mathbb{E}\left[\int_0^T e^{\beta r}\left(\left|\hat{Y}_r\right|^2 + \left|\hat{Z}_r\right|^2\right) dr\right] \le \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta r}\left(\left|\hat{y}_r\right|^2 + \left|\hat{z}_r\right|^2\right) dr\right],$$

that is, $\left\|\left(\hat{Y}, \hat{Z}\right)\right\|_\beta \le \frac{1}{\sqrt{2}}\left\|(\hat{y}, \hat{z})\right\|_\beta$. Consequently, I is a contraction on $\mathcal{H}_{\mathbb{F}}^2\left(0,T;\mathbb{R}\times\mathbb{R}^d\right)$ endowed with the norm $\left\|\cdot\right\|_\beta$, and from the contraction mapping theorem we know that there is a unique fixed point $(Y,Z) \in \mathcal{H}_{\mathbb{F}}^{2}\left(0,T;\mathbb{R}\times\mathbb{R}^{d}\right)$ such that $I(Y,Z)=(Y,Z)$. On the other hand, from Step 01 we already know that if $I(Y,Z)=(Y,Z)$ then

$$(Y,Z) \in \mathcal{S}_{\mathbb{F}}^{2}(0,T;\mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^{2}\left(0,T;\mathbb{R}^{d}\right). \quad \blacksquare$$

Chapter 2

Backward doubly stochastic differential equations

In this chapter , we recall some results on BDSDEs, we'll present the existence and uniqueness of solutions for BDSDEs, under uniformly Lipschitz condition and estimate the moments of the solution. For this and, we present the comparison theorem of BDSDEs because we know, its very useful result in the theory of BDSDEs. Then we give the proof the existence of minimal (resp maximal) solutions for the BDSDEs with continuous coefficients. Next we recall the result about the reflected BDSDEs with Lipschitz condition. Finaly we present the proof of the existence of a minimal solustion in the case when the coefficient f is continuous and with linear growth.

2.1 Backward doubly stochastic differential equations with Lipschitz coefficients

2.1.1 Notation and assumptions

Let T be a fixed final time. Throughout this thesis $\{W_t,\ 0 \leq t \leq T\}$ and $\{B_t,\ 0 \leq t \leq T\}$ will denote two independent d-dimensional Brownian motions ($d \geq 1$), defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \triangleq \mathcal{F}_t^W \otimes \mathcal{F}_{t,T}^B \vee \mathcal{N}\ , \text{ and } \mathcal{G}_t = \mathcal{F}_t^W \otimes \mathcal{F}_T^B,$$

where $\mathcal{F}_t^W = \sigma\left(W_s : 0 \leq s \leq t\right)$ and $.\mathcal{F}_{t,T}^B = \sigma\left(B_s - B_t : t \leq s \leq T\right)$.

In other words the σ-fields $\mathcal{F}_t$, $0 \leq t \leq T$, are $\mathbb{P}$-complete. We notice that the family of σ-algebras $\mathrm{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is neither increasing nor decreasing; in particular, it is not a filtration.

We consider coefficients (f, g) with the following properties:

$$f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \longrightarrow \mathbb{R}^n,$$

$$g : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \longrightarrow \mathbb{R}^{n \times d},$$

be jointly measurable and such that for any $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$,

$$f\left(., y, z\right) \in \mathcal{H}^2\left([0, T]; \mathbb{R}^n\right),$$

$$g\left(., y, z\right) \in \mathcal{H}^2\left([0, T]; \mathbb{R}^n\right),$$

the following hypotheses are satisfied for some strictly positive finite constant C and $0 < \alpha < 1$ such that for any $(\omega, t) \in \Omega \times [0, T]$, (y_1, z_1), $(y_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$, :

$$(\mathrm{H}_2) \qquad \begin{cases} f\,(t, y, z)\ ,\ g\,(t, y, z) \text{ are } \mathcal{F}_t\text{-measurable processes}\,, \\ \text{(i)}\ \ |f\,(t, y_1, z_1) - f\,(t, y_2, z_2)\,|^2 \le c\,\left(|y_1 - y_2|^2 + \|z_1 - z_2\|^2\right), \\ \text{(ii)}\ \ |g\,(t, y_1, z_1) - g\,(t, y_2, z_2)\,|^2 \le c\,|y_1 - y_2|^2 + \alpha \|z_1 - z_2\|^2. \end{cases}$$

Throughout this paper, $<,>$ will denote the scalar product on $\mathbb{R}^n$, i.e $< x, y > := \sum_{i=1}^n x_i y_i$, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Sometimes, we will also use the notation $x^* y$ to designate $< x, y >$. We point out that by C we always denote a finite constant whose value may change from one line to the next, and which usually is (strictly) positive.

2.1.2 Existence and uniqueness of a Solution

Suppose that we are given a terminal condition $\xi \in \mathbb{L}^2\,(\Omega, \mathcal{F}_T, \mathbb{P})$. The solution to a **BDSDE**(ξ, f, g) is a pair $(Y, Z) \in \mathcal{S}^2\,([0, T];\ \mathbb{R}^n) \times \mathcal{H}^2\left([0, T]; \mathbb{R}^{n\times d}\right)$, such that for any $0 \le t \le T$

$$Y_t = \xi + \int_t^T f\,(s, Y_s, Z_s)\,ds + \int_t^T g\,(s, Y_s, Z_s)\,\overleftarrow{dB_s} - \int_t^T Z_s dW_s. \tag{2.1}$$

Here $\overleftarrow{dB_s}$ denotes the classical backward Itô integral with respect to the Brownian motion B, and dW denotes the standard forward Itô integral with respect to the Brownian motion W. Our main goal in this section is to prove the following theorem.

Theorem 11 *Under the above conditions, in particular* (H_2), *Eq* (2.1) *has unique solution*

$$(Y, Z) \in \mathcal{S}^2\,([0, T];\ \mathbb{R}^n) \times \mathcal{H}^2\left([0, T]; \mathbb{R}^{n\times d}\right).$$

Let us first establish the result in Theorem (11) for BDSDEs, where the coefficients f, g do not depend on Y and Z. Given $f \in \mathcal{H}^2\left([0,T];\mathbb{R}^{n\times d}\right)$ and $g \in \mathcal{H}^2\left([0,T];\mathbb{R}^{n\times d}\right)$, and let ξ be as before. Consider the equation:

$$Y_t = \xi + \int_t^T f(s)\,ds + \int_t^T g(s)\,\overleftarrow{dB_s} - \int_t^T Z_s dW_s. \tag{2.2}$$

Then we have the following result. There exists a unique pair

$$(Y,Z) \in \mathcal{S}^2\left([0,T];\ \mathbb{R}^n\right) \times \mathcal{H}^2\left([0,T];\mathbb{R}^{n\times d}\right).$$

which solves Eq(2.2).

Proof. Existence. To show the existence, we consider the filtration $\mathcal{G}_t = \mathcal{F}_t^W \otimes \mathcal{F}_T^B$ and the martingale

$$M_t = \mathbb{E}\left[\xi + \int_0^T f(s)\ ds + \int_0^T g(s)\,\overleftarrow{dB_s}/\mathcal{G}_t\right], \tag{2.3}$$

which is clearly a square integrable martingale by (H_2). An extension of Itô's martingale representation theorem yields the existence of a $\mathcal{G}_t$-progressively measurable process (Z_t) with values in $\mathbb{R}^{n\times d}$ such that

$$\mathbb{E}\int_0^T \|Z_t\|^2 dt < \infty \quad \text{and} \quad M_T = M_t + \int_t^T Z_s dW_s,\ t \in [0,t] \tag{2.4}$$

We subtract the quantity $\int_0^t f(s)\,ds + \int_0^t g(s)\,\overleftarrow{dB_s}$ from both sides of the martingale in (2.3) and we employ the martingale representation in (2.4) to obtain

$$Y_t = \xi + \int_t^T f(s)\,ds + \int_t^T g(s)\,\overleftarrow{dB_s} - \int_t^T Z_s dW_s,$$

where

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f(s)\,ds + \int_t^T g(s)\,\overleftarrow{dB_s}/\mathcal{G}_t\right].$$

It remains to show that (Y_t) and (Z_t) are in fact $\mathcal{F}_t$-adapted. For Y_t, this is obvious since for each t,

$$Y_t = \mathbb{E}(\Theta/\mathcal{F}_t \vee \mathcal{F}_t^B)$$

Where Θ is $\mathcal{F}_T \vee \mathcal{F}_{t,T}^B$ measurable. Hence $\mathcal{F}_t^B$ is independent of $\mathcal{F}_t \vee \sigma(\Theta)$, and

$$Y_t = \mathbb{E}(\Theta/\mathcal{F}_t).$$

Now

$$\int_t^T Z_s dW_s = \xi + \int_t^T f(s)\, ds + \int_t^T g(s)\, \overleftarrow{dB_s} - Y_t,$$

and the right side is $\mathcal{F}_T^W \vee \mathcal{F}_{t,T}^B$ measurable. Hence, from Itô's martingale representation theorem, $Z_s, t < s < T$ is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ adapted. Consequently Z_s is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ measurable, for any $t < s$, so it is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ measurable.

Uniqueness. Is immediate, since if $(\overline{Y}, \overline{Z})$ is the difference of two solutions,

$$\overline{Y_t} + \int_t^T \overline{Z_s} dW_s = 0, \quad 0 \le t \le T.$$

Hence by orthogonality

$$\mathbb{E}(|\overline{Y_t}|^2) + \mathbb{E}\int_t^T Tr[\overline{Z_s Z_s^*}]ds = 0,$$

and $\overline{Y_t} \equiv 0\ Pa.s.,\ \overline{Z_t} = 0\ dtdPa.e.$ ∎

We will also need the following Itô-formula.

Lemma 12 *Let* $\alpha \in \mathcal{S}^2(\mathrm{F}, [0,T]; \mathbb{R}^n)$, $\beta \in \mathcal{H}^2(\mathrm{F}, [0,T]; \mathbb{R}^n)$, $\gamma \in \mathcal{H}^2\left(\mathrm{F}, [0,T]; \mathbb{R}^{n\times d}\right)$, *and* $\delta \in \mathcal{H}^2\left(\mathrm{F}, [0,T];\ \mathbb{R}^{n\times d}\right)$ *be such that:*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s \overleftarrow{dB_s} + \int_0^t \delta_s dW_s.$$

Then, for any function $\phi \in \mathcal{C}^2(\mathbb{R}^n,)$

$$\phi(\alpha_t) = \phi(\alpha_0) + \int_0^t \langle \nabla\phi(\alpha_s), \beta_s\rangle ds + \int_0^t \left\langle \nabla\phi(\alpha_s), \gamma_s \overleftarrow{dB_s}\right\rangle + \int_0^t \langle \nabla\phi(\alpha_s), \delta_s dW_s\rangle - \frac{1}{2}\int_0^t \mathrm{Tr}[\phi''(\alpha_s)\ \gamma_s\,\gamma_s^*]\, ds + \frac{1}{2}\int_0^t \mathrm{Tr}[\phi''(\alpha_s)\ \delta_s\,\delta_s^*]\, ds.$$

In particular,

$$|\alpha|_t^2 = |\alpha_0|^2 + 2\int_0^t \langle \alpha_s, \beta_s\rangle ds + 2\int_0^t \left\langle \alpha_s, \gamma_s \overleftarrow{dB_s}\right\rangle + 2\int_0^t \langle \alpha_s, \delta_s dW_s\rangle - \int_0^t \|\gamma_s\|^2 ds + \int_0^t \|\delta_s\|^2 ds.$$

Next, we establish a prior estimate for the solution of the BSDE in (2.1). for that sake, we need an additional assumption on g.

$$(\mathrm{H}_3) \qquad \begin{cases} \texttt{there exists } c \texttt{ such that for all}(t,y,z) \in [0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}, \\ gg^*(t,y,z) \le zz^* + c(\|g(t,0,0)\|^2 + |y|^2)I. \end{cases}$$

Proposition 13 *Assume, in addition to the condition of Theorem(11), that (H3) holds and for some* $p > 2$, $\xi \in L^p(\Omega, F_T, P, \mathbb{R}^k)$ *and*

$$E\int_0^T (|f(t,0,0)|^p + \|g(t,0,0)\|^p)dt < \infty.$$

Then

$$E(\sup_{0\le t\le T} |Y_t|^p + (\int_0^T \|Z_t\|^2)^{p/2}) < \infty.$$

Proof. By lemma 12 applied to $\varphi(x) = |x|^p$, we obtain that

$$|Y_t|^p + \frac{p}{2}\int_t^T |Y_s|^{p-2}\|Z_s\|^2 ds + \frac{p}{2}(p-2)\int_t^T |Y_s|^{p-4}(Z_sZ_s^*Y_s, Y_s)ds$$
$$= |\xi|^p + p\int_t^T |Y_s|^{p-2}\langle f(s,Y_s,Z_s), Y_s\rangle ds + p\int_t^T |Y_s|^{p-2}\langle Y_s, g(s,Y_s,Z_s)dB_s\rangle$$
$$+ \frac{p}{2}\int_t^T |Y_s|^{p-2}\|g(s,Y_s,Z_s)\|^2 ds$$
$$+ \frac{p}{2}(p-2)\int_t^T |Y_s|^{p-4}\langle gg^*(s,Y_s,Z_s)Y_s, Y_s\rangle ds - p\int_t^T |Y_s|^{p-2}\langle Y_s, Z_s dW_s\rangle.$$

Taking the expectation, we get

$$\mathbb{E}(|Y_t|^p) + \frac{p}{2}\mathbb{E}\int_t^T |Y_s|^{p-2}\|Z_s\|^2 ds + \frac{p}{2}(p-2)\mathbb{E}\int_t^T |Y_s|^{p-4}\langle Z_s Z_s^* Y_s, Y_s\rangle ds$$
$$\leq \mathbb{E}(|\xi|^p) + p\mathbb{E}\int_t^T |Y_s|^{p-2}\langle f\left(s, Y_s, Z_s\right), Y_s\rangle ds + \frac{p}{2}\mathbb{E}\int_t^T |Y_s|^{p-2}\|g\left(s, Y_s, Z_s\right)\|^2 ds$$
$$+ \frac{p}{2}(p-2)\mathbb{E}\int_t^T |Y_s|^{p-4}\langle gg^*(s, Y_s, Z_s)Y_s, Y_s\rangle ds.$$

We can conclude from (H_2) that for any $\alpha < \alpha' < 1$, there exists $c(\alpha')$ such that for $0 \leq t \leq T$,

$$\| g(t, y, z) \| \leq c\left(\alpha'\right)(|y|^2 + \| g(t, 0, 0) \|^2) + \alpha' \|z\|^2 .$$

But from (H_2), (H_3) and the fact that $2\,ab \leq \dfrac{1-\alpha}{2c}a^2 + \dfrac{2c}{1-\alpha}b^2$, $c > 0$, it follows that there exists a constant $\theta > 0$ and c such that

$$\mathbb{E}(|Y_t|^p) + \theta\mathbb{E}\int_t^T |Y_s|^{p-2}\|Z_s\|^2 ds$$
$$\leq \mathbb{E}(|\xi|^p) + c\mathbb{E}\int_t^T (|Y_s|^p + |f\left(s, 0, 0\right)|^p + \|g\left(s, 0, 0\right)\|^p) ds$$

Then, from Gronwall‘s Lemma we obtain

$$\sup_{0\leq t\leq T} \mathbb{E}\left(|Y_t|^p + \int_0^T |Y_t|^{p-2}\|Z_t\|^2 dt\right) < \infty$$

Applying the same inequalities we have already used to the first identity of the proof, we deduce that

$$|Y_t|^p \leq |\xi|^p + c\int_t^T (|Y_s|^p + |f(s, 0, 0)|^p + \|g(s, 0, 0)\|^p) ds$$
$$+ p\int_t^T |Y_s|^{p-2}\langle Y_s, g(s, Y_s, Z_s)dB_s\rangle - p\int_t^T |Y_s|^{p-2}\langle Y_s, Z_s dW_s\rangle$$

from the Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}(sup_{0\le t\le T}|Y_t|^p) \le \mathbb{E}|\xi|^p + c\mathbb{E}\int_0^T (|Y_s|^p + |f(s,0,0)|^p + \|g(s,0,0)\|^p)ds$$
$$+ c\mathbb{E}\sqrt{\int_0^T |Y_s|^{2p-4}\langle gg^*(s,Y_s,Z_s)Y_s,Y_s\rangle ds}$$
$$+ c\mathbb{E}\sqrt{\int_0^T |Y_s|^{2p-4}\langle Z_sZ_s^*Y_s,Y_s\rangle ds}$$

We estimate the last term as follows :

$$\mathbb{E}\sqrt{\int_0^T |Y_s|^{2p-4}\langle Z_sZ_s^*Y_s,Y_s\rangle ds} \le \mathbb{E}(Y_t^{p/2}\sqrt{\int_0^T |Y_t|^{p-2}\|Z_t\|^2dt}$$
$$\le \frac{1}{3}\mathbb{E}(sup_{0\le t\le T}|Y_t|^p) + \frac{1}{4}\mathbb{E}\int_0^T |Y_t|^{p-2}\|Z_t\|^2dt$$

we deduce that

$$\mathbb{E}(sup_{0\le t\le T}|Y_t|^p) < \infty$$

Now we have

$$\int_0^T \|Z_t\|^2dt = |\xi|^2 - |Y_0|^2 + 2\int_0^T \langle f(t,Y_t,Z_t),Y_t\rangle dt + 2\int_0^T \langle Y_t, g(t,Y_t,Z_t)dB_t\rangle$$
$$+ \int_0^T \|g(t,Y_t,Z_t)\|^2dt - 2\int_0^T \langle Y_t, Z_tdW_t\rangle$$

Hence for any $\delta > 0$,

$$(\int_0^T \|Z_t\|^2dt)^{p/2} \le (1+\delta)(\int_0^T \|g(t,Y_t,Z_t)\|^2dt)^{p/2} + c(\delta,p)[|\xi|^p + |Y_0|^p + |\int_0^T \langle f(t,Y_t,Z_t),Y_t\rangle dt|^{p/2}$$
$$+ |\int_0^T \langle Y_t, g(t,Y_t,Z_t)dB_t\rangle|^{p/2} + |\int_0^T \langle Y_t, Z_tdW_t\rangle|^{p/2}]$$

Passing to expectation

$$\begin{aligned}
&\mathbb{E}(\int_0^T \|Z_t\|^2 dt)^{p/2} \leq (1+\delta)^2\alpha\mathbb{E}(\int_0^T \|Z_t\|^2 dt)^{p/2} + c'(\delta,p) \\
&+ c(\delta,p)\mathbb{E}[\left(\int_0^T \|Z_t\||Y_t|dt\right)^{p/2}] + c(\delta,p)\mathbb{E}[(\int_0^T |Y_t|^2\|Z_t\|^2 dt)^{p/4}] \\
&\leq (1+\delta)^2\alpha\mathbb{E}(\int_0^T \|Z_t\|^2 dt)^{p/2} + c'(\delta,p) \\
&+ c(\delta,p)\mathbb{E}\{(sup_{0\leq t\leq T}|Y_t|^{p/2})[(\int_0^T \|Z_t\|dt)^{p/2} + (\int_0^T \|Z_t\|^2 dt)^{p/4}]\} \\
&\leq [(1+\delta)^2\alpha + (1+\delta)]\mathbb{E}[(\int_0^T \|Z_t\|^2 dt)^{p/2}] + c"(\delta,p).
\end{aligned}$$

The second part of the result now follows, if we choose $\delta > 0$ small enough such that

$$(1+\delta)^2\alpha + (1+\delta) < 1$$

■

We can now turn to the proof of theorem (11)

Proof. Uniqueness. Let (Y_t^1, Z_t^1) and (Y_t^2, Z_t^2) be two solutions. Define

$$\overline{Y}_t = Y_t^1 - Y_t^2, \quad \overline{Z}_t = Z_t^1 - Z_t^2, \quad 0 \leq t \leq T$$

Then

$$\overline{Y}_t = \int_t^T [f(s,Y_s^1,Z_s^1) - f(s,Y_s^2,Z_s^2)]ds + \int_t^T [g(s,Y_s^1,Z_s^1) - g(s,Y_s^2,Z_s^2)]dB_s - \int_t^T \overline{Z}_s dW_s.$$

Applying Itô's formula to $|\overline{Y}_t|^2$ yields :

$$\begin{aligned}
\mathbb{E}|\overline{Y}_t|^2 + \mathbb{E}\int_t^T \|\overline{Z}_s\|^2 ds &= 2\mathbb{E}\int_t^T \langle f(s,Y_s^1,Z_s^1) - f(s,Y_s^2,Z_s^2), \overline{Y}_s\rangle ds \\
&+ \mathbb{E}\int_t^T \|g(s,Y_s^1,Z_s^1) - g(s,Y_s^2,Z_s^2)\|^2 ds.
\end{aligned}$$

Hence from (H$_2$) and the inequality $ab \leq \frac{1}{2(1-\alpha)}a^2 + \frac{1-\alpha}{2}b^2$,

$$\mathbb{E}|\overline{Y}_t|^2 + \mathbb{E}\int_t^T \|\overline{Z}_s\|^2 ds \leq c(\alpha)\mathbb{E}\int_t^T |\overline{Y}_s|^2 ds + \frac{1-\alpha}{2}\mathbb{E}\int_t^T \left\|\overline{Z}_s\right\|^2 ds + \alpha\mathbb{E}\int_t^T \left\|\overline{Z}_s\right\|^2 ds.$$

where $0 < \alpha < 1$ is the constant appearing in (H$_2$). Consequently

$$\mathbb{E}|\overline{Y}_t|^2 + \frac{1-\alpha}{2}\mathbb{E}\int_t^T \|\overline{Z}_s\|^2 ds \leq c(\alpha)\mathbb{E}\int_t^T |\overline{Y}_s|^2 ds.$$

From Gronwall's lemma, $\mathbb{E}(|\overline{Y}_t|^2) = 0, \quad 0 \leq t \leq T$, and hence $\mathbb{E}\int_0^T \|\overline{Z}_s\|^2 = 0$.

Existence. We define recursively a sequence $(Y_t^n, Z_t^n)_{n=0,1,\ldots}$ as follows. Let $Y_t^0 \equiv 0,\ Z_t^0 \equiv 0$. Given (Y_t^n, Z_t^n), (Y_t^{n+1}, Z_t^{n+1}) is the unique solution, constructed as in theorem (11), of the following equation :

$$Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + \int_t^T g(s, Y_s^n, Z_s^n)dB_s - \int_t^T Z_s^{n+1}dW_s. \tag{2.5}$$

Let $\overline{Y}_t^{n+1} \triangleq Y_t^{n+1} - Y_t^n,\ \overline{Z}_t^{n+1} \triangleq Z_t^{n+1} - Z_t^n,\ 0 \leq t \leq T$. The same computation as in the proof of uniqueness yield :

$$E(|\overline{Y}_t^{n+1}|^2) + \mathbb{E}\int_t^T \|\overline{Z}_t^{n+1}\|^2 ds = 2\mathbb{E}\int_t^T \langle f(s, Y_s^n, Z_s^n) - f(s, Y_s^{n-1}, Z_s^{n-1}), \overline{Y}_t^{n+1}\rangle$$
$$+ \mathbb{E}\int_t^T \|g(s, Y_s^n, Z_s^n) - g(s, Y_s^{n-1}, Z_s^{n-1})\|^2 ds.$$

Let $\beta \in \mathbb{R}$. By integration by parts, we deduce

$$\mathbb{E}(|\overline{Y}_t^{n+1}|^2 e^{\beta t}) + \beta\mathbb{E}\int_t^T |\overline{Y}_t^{n+1}|^2 e^{\beta s} ds + \mathbb{E}\int_t^T \|\overline{Z}_t^{n+1}\|^2 e^{\beta s} ds$$
$$= 2\mathbb{E}\int_t^T \langle f(s, Y_s^n, Z_s^n) - f(s, Y_s^{n-1}, Z_s^{n-1}), \overline{Y}_t^{n+1}\rangle e^{\beta s} ds$$
$$+ \mathbb{E}\int_t^T \|g(s, Y_s^n, Z_s^n) - g(s, Y_s^{n-1}, Z_s^{n-1})\|^2 e^{\beta s} ds.$$

There exists $c, \gamma > 0$ such that

$$\mathbb{E}(|\overline{Y}_t^{n+1}|^2 e^{\beta t}) + (\beta - \gamma)\mathbb{E}\int_t^T |\overline{Y}_t^{n+1}|^2 e^{\beta s} ds + \mathbb{E}\int_t^T \|\overline{Z}_t^{n+1}\|^2 e^{\beta s} ds$$
$$\leq \mathbb{E}\int_t^T (c|\overline{Y}_s^n|^2 + \frac{1+\alpha}{2}\|Z_s^n\|^2) e^{\beta s} ds.$$

Now choose $\beta = \gamma + \frac{2c}{1+\alpha}$, and define $\overline{c} = \frac{2c}{1+\alpha}$,

$$\mathbb{E}(|\overline{Y}_t^{n+1}|^2 e^{\beta t}) + \mathbb{E}\int_t^T (\overline{c}|\overline{Y}_t^{n+1}|^2 + \|\overline{Z}_t^{n+1}\|^2) e^{\beta s} ds$$
$$\leq \frac{1+\alpha}{2}\mathbb{E}\int_t^T (\overline{c}|\overline{Y}_s^n|^2 + \|Z_s^n\|^2) e^{\beta s} ds.$$

It follows immediately that

$$\mathbb{E}\int_t^T (\overline{c}|\overline{Y}_t^{n+1}|^2 + \|\overline{Z}_t^{n+1}\|^2) e^{\beta s} ds \leq (\frac{1+\alpha}{2})^n \mathbb{E}\int_t^T (\overline{c}|\overline{Y}_s^1|^2 + \|Z_s^1\|^2) e^{\beta s} ds.$$

and, since $\frac{1+\alpha}{2} < 1$, $(Y_t^n, Z_t^n)_{n=0,1,\ldots}$ is a Cauchy sequence in $\mathcal{H}^2(0,T;\mathbb{R}^k) \times \mathcal{H}^2(0,T;\mathbb{R}^{k\times l})$. It is then easy to conclude $(Y_t^n)_{n=0,1,\ldots}$ is also Cauchy in $S^2([0,T];\mathbb{R}^k)$, and that

$$(Y_t, Z_t) = lim_{n\to\infty}(Y_t^n, Z_t^n)$$

solves equation (2.1). ■

2.2 Comparison Theorem of Backward doubly stochastic differential equations

In this section, we only consider one-dimensional BDSDEs. We consider the following BDSDEs : $(0 \leq t \leq T)$

$$Y_t^1 = \xi^1 + \int_t^T f^1(s, Y_s^1, Z_s^1) ds + \int_t^T g(s, Y_s^1, Z_s^1) dB_s - \int_t^T Z_s^1 dW_s \quad (2.6)$$
$$Y_t^2 = \xi^2 + \int_t^T f^2(s, Y_s^2, Z_s^2) ds + \int_t^T g(s, Y_s^2, Z_s^2) dB_s - \int_t^T Z_s^2 dW_s \quad (2.7)$$

where BDSDEs (2.6) and (2.7) satisfy the conditions of theorem (11). Then there exist two pairs of measurable processes (Y^1, Z^1) and (Y^2, Z^2) satisfying BDSDEs (2.6) and (2.7), respectively. Assume

$$(\mathrm{H}_4) \qquad \begin{cases} \xi^1 \geq \xi^2, \quad a.s., \\ f^1(t,Y,Z) \geq f^2(t,Y,Z), \quad a.s., \end{cases}$$

Then we have the following comparison theorem.

Theorem 14 *Assume BDSDEs (2.6) and (2.7) satisfy the conditions of theorem (11), let (Y^1, Z^1) and (Y^2, Z^2) be solutions of BDSDEs (2.6) and (2.7), respectively. If (H4) holds, then $Y_t^1 \geq Y_t^2,\ a.s,\ \forall t \in [0,T]$.*

Proof. The pair $(Y_t^1 - Y_t^2, Z_t^1 - Z_t^2)$ satisfies the following BDSDE.

$$\begin{aligned} Y_t^1 - Y_t^2 = (\xi^1 - \xi^2) &+ \int_t^T (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2))ds \\ &+ \int_t^T (g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2))dB_s \\ &- \int_t^T (Z_s^1 - Z_s^2)dW_s, \quad 0 \leq t \leq T. \end{aligned}$$

Applying Itô's formula to $|(Y_t^1 - Y_t^2)^-|^2$, we get

$$\begin{aligned} |(Y_t^1 - Y_t^2)^-|^2 = |(\xi^1 - \xi^2)^-|^2 &- 2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2))ds \\ &- 2\int_t^T (Y_s^1 - Y_s^2)^-(g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2))dB_s \\ &+ \int_t^T 1_{[Y_s^1 \leq Y_s^2]}|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)|^2 ds \\ &+ 2\int_t^T (Y_s^1 - Y_s^2)^-(Z_s^1 - Z_s^2)dW_s - \int_t^T 1_{[Y_s^1 \leq Y_s^2]}|Z_s^1 - Z_s^2|^2 ds. \end{aligned} \tag{2.8}$$

From (H_4), we have $\xi^1 - \xi^2 \geq 0$, so

$$\mathbb{E}|(\xi^1 - \xi^2)^-|^2 = 0.$$

Since (Y^1, Z^1) and (Y^2, Z^2) are in $\mathcal{S}^2([0,T];\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R}^d)$ it easily follows that

$$\mathbb{E}\int_t^T (Y_s^1 - Y_s^2)^-(Z_s^1 - Z_s^2)dW_s = 0,$$

$$\mathbb{E}\int_t^T (Y_s^1 - Y_s^2)^-(g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2))dB_s = 0.$$

Let

$$\begin{aligned}
\Delta &= -2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2))ds \\
&= -2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2))ds \\
&- 2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2))ds \\
&= \Delta_1 + \Delta_2,
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1 &= -2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2))ds \\
\Delta_2 &= -2\int_t^T (Y_s^1 - Y_s^2)^-(f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2))ds \leq 0.
\end{aligned}$$

From (H_2) and Young's inequality, it follows that

$$\begin{aligned}
\Delta \leq \Delta_1 &\leq 2C\int_t^T (Y_s^1 - Y_s^2)^-(|Y_s^1 - Y_s^2| + |Z_s^1 - Z_s^2|)ds \\
&\leq (2C + \frac{C^2}{1-\alpha})\int_t^T |(Y_s^1 - Y_s^2)^-|^2 ds + (1-\alpha)\int_t^T 1_{[Y_s^1 \leq Y_s^2]}|Z_s^1 - Z_s^2|^2 ds,
\end{aligned}$$

Using the assumption (H_2), again, we deduce

$$\begin{aligned}
&\int_t^T 1_{[Y_s^1 \leq Y_s^2]}|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)|^2 ds \\
&\leq \int_t^T 1_{[Y_s^1 \leq Y_s^2]}[C|Y_s^1 - Y_s^2|^2 + \alpha|Z_s^1 - Z_s^2|^2]ds \\
&= C\int_t^T |(Y_s^1 - Y_s^2)^-|^2 ds + \alpha\int_t^T 1_{[Y_s^1 \leq Y_s^2]}|Z_s^1 - Z_s^2|^2 ds.
\end{aligned}$$

Taking expectation on both sides of (2.8), we get

$$\mathbb{E}|(Y_t^1 - Y_t^2)^-|^2 \leq (C + 2C + \frac{C^2}{1-\alpha})\mathbb{E}\int_t^T |(Y_s^1 - Y_s^2)^-|^2 ds.$$

By Gronwall's inequality, it follows that

$$\mathbb{E}|(Y_t^1 - Y_t^2)^-|^2 = 0 \quad \forall t \in [0,T].$$

That is, $Y_t^1 \geq Y_t^2$, $a.s.$, $\forall t \in [0,T]$. ■

2.3 Backward doubly stochastic differential equations with continuous coefficient

In this section we study BDSDEs with continuous coefficient. Our main result is We consider coefficients (f, g) with the following properties:

$$f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R},$$

$$g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^l,$$

for $a.e$ (t,ω), the map (y,z), the map $(y,z) \longmapsto f(t,y,z)$ is continuous, and for any $(t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d$, the following hypotheses are satisfied for some strictly positive finite constant K, L and $0 < \alpha < 1$ such that for any (y_1, z_1), $(y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$,

$$(H_5) \qquad \begin{cases} f\,(t,y,z)\;,\; g\,(t,y,z) \text{ are } \mathcal{F}_t\text{-measurable processes}\,, \\ \text{(i)} \;\; |f\,(t,y,z)\,| \leq K\;(1 + |y| + |z|)\,, \\ \text{(ii)} \;\; |g\,(t,y_1,z_1) - g\,(t,y_2,z_2)\,|^2 \leq c\,|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2. \end{cases}$$

Theorem 15 *Under the above hypothese (H_5) and if $\xi \in L^2$, there exists a solution for the BDSDE (2.1). Moreover, there is a minimal solution $(\underline{Y}, \underline{Z})$ of BDSDE (2.1) in the sense that, for any other solution (Y, Z) of BDSDE (2.1), we have $\underline{Y} \leq Y$.*

We still assume that $l = d = 1$. Before giving the proof of Theorem 15, we define,as the classical approximation can be proved by adapting the proof given in J. J. Alibert and K. Bahlali [1], the sequence $f_n(t, y, z)$ associated to f,

$$f_n(t, y, z) = \inf_{y', z' \in \mathcal{Q}} [f(t, y', z') + n(|y - y'| + |z - z'|)],$$

then for $n \geq N$, f_n is jointly measurable and uniformly linear growth in y, z with constant N. We also define the function.

$$F(t, y, z) = N(1 + |y| + |Z|)$$

Given $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, by theorem (11), there exist two pair of processes (Y^n, Z^n) and (U, V), which are the solutions to the following BDSDEs, respectively,

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n) ds + \int_t^T g(s, Y_s^n, Z_s^n) dB_s - \int_t^T Z_s^n dW_s \tag{2.9}$$

$$U_t = \xi + \int_t^T F(s, U_s, V_s) ds + \int_t^T g(s, U_s, V_s) dB_s - \int_t^T V_s dW_s \tag{2.10}$$

From theorem (14) and lemma 1 of [34], we get

$$\forall\ n \geq m \geq N,\ Y^m \leq Y^n \leq U,\ dt \otimes dP - a.s. \tag{2.11}$$

Lemma 16 *There exists a constant $A > 0$ depending only on N, C, α, T and ξ, such that*

$$\forall\ n \geq N, \|Y^n\| \leq A,\ \|Z^n\| \leq A;\ \|U\| \leq A,\ \|V\| \leq A$$

.

The proof of this lemma is in [45]

Lemma 17 $\{(Y^n, Z^n)\}_{n=1}^{+\infty}$ *converges in* $\mathcal{S}^2([0,T];\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R})$.

Proof. Let $n_0 \geq N$. Since $\{y^n\}$ is increasing and bounded in $\mathcal{S}^2([0,T];\mathbb{R})$, we deduce from the dominated convergence theorem that Y^n converges in $\mathcal{S}^2([0,T];\mathbb{R})$. We shall denote by Y the limit of $\{Y^n\}$. Applying Itô's formula to $|Y_t^n - Y_t^m|^2$, we get for n,m$\geq n_0$;

$$\begin{aligned}
&\mathbb{E}\left|Y_0^n - Y_0^m\right|^2 + \mathbb{E}\int_0^T \left|Z_s^n - Z_s^m\right|^2 ds \\
&= 2\mathbb{E}\int_0^T \left(Y_s^n - Y_s^m\right)\left(f_n\left(s, Y_s^n, Z_s^n\right) - f_m\left(s, Y_s^m, Z_s^m\right)\right) ds \\
&+\mathbb{E}\int_0^T \left|g\left(s, Y_s^n, Z_s^n\right) - g\left(s, Y_s^m, Z_s^m\right)\right|^2 ds \\
&\leq 2\left(\mathbb{E}\int_0^T \left|Y_s^n - Y_s^m\right|^2 ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T \left|f_n\left(s, Y_s^n, Z_s^n\right) - f_m\left(s, Y_s^m, Z_s^m\right)\right|^2 ds\right)^{\frac{1}{2}} \\
&+\mathbb{E}\int_0^T \left(C\left|Y_s^n - Y_s^m\right|^2 + \alpha\left|Z_s^n - Z_s^m\right|^2\right) ds.
\end{aligned}$$

Since f_n and f_m are uniformly linear growth and $\{Y^n, Z^n\}$ is bounded, similary to lemma (16), there exists a constant $\bar{N} > 0$ depending only on N, C, α, T and ξ, such that

$$\begin{aligned}
&\mathbb{E}\left|Y_0^n - Y_0^m\right|^2 + \mathbb{E}\int_0^T \left|Z_s^n - Z_s^m\right|^2 ds \\
&\leq \mathbb{E}\left(\int_0^T \bar{N}\left|Y_s^n - Y_s^m\right|^2 + \alpha\left|Z_s^n - Z_s^m\right|^2 ds\right).
\end{aligned}$$

So

$$\|Z^n - Z^m\|^2 \leq \frac{\bar{N}T}{1-\alpha}\|Y^n - Y^m\|^2.$$

Thus $\{Z^n\}$ is a Cauchy sequence in $\mathcal{H}^2(0,T;\mathbb{R})$, from which the result follows. ■

Proof. of Theorem(15). For all $n \geq n_0 \geq N$, we have $Y^{n_0} \leq Y^n \leq U$, and $\{Y^n\}$ converges in $\mathcal{H}^2(0,T;\mathbb{R})$, $dt \otimes d\mathbb{P}-a.s.$ to $Y \in \mathcal{S}^2([0,T];\mathbb{R})$.

On the other hand, since Z^n converges in $\mathcal{H}^2(0,T;\mathbb{R})$ to Z, we can assume, choosing a subsequence if needed, that $Z^n \to Z, dt \otimes d\mathbb{P} - a.s.$ and $\bar{G} = \sup_n |Z^n|$ is $d\mathbb{P} - a.s.$integrable. Therefore, from (i) and (iv) of lemma 1 in [34], we get

$$f_n(t, Y_t^n, Z_t^n) \to f(t, Y, Z), \quad (n \to \infty)\, dt{-}a.s.$$

$$\begin{aligned} |f_n(t, Y_t^n, Z_t^n)|\, ds &\leq N(1 + \sup_n |Y_t^n| + \sup_n |Z_t^n|) \\ &= N\left(1 + \sup_n |Y_t^n| + \bar{G}_t\right) \in L^1([0,T], dt). \end{aligned}$$

Thus, for almost all ω and uniformly in t, it holds that

$$\int_t^T f_n(s, Y_s^n, Z_s^n)\, ds \to \int_t^T f(s, Y, Z)\, ds, \quad (n \to \infty).$$

From the continuity properties of the stochastic integral, it follows that

$$\sup_{0 \leq t \leq T} \left| \int_t^T Z_s^n dW_s - \int_t^T Z_s dW_s \right| \to 0 \text{ in probability},$$

$$\sup_{0 \leq t \leq T} \left| \int_t^T g(s, Y_s^n, Z_s^n)\, dB_s - \int_t^T g(s, Y_s, Z_s)\, dB_s \right| \to 0 \text{ in probability}.$$

Choosing, again a subsequence, we can assume that the above convergence is $\mathbb{P} - a.s.$ Finally,

$$\begin{aligned} \left|Y_t^n - Y_t^m\right| & \leq \int_t^T \left|f_n\left(s, Y_s^n, Z_s^n\right) - f_m\left(s, Y_s^m, Z_s^m\right)\right| ds \\ & + \left|\int_t^T g\left(s, Y_s^n, Z_s^n\right) dB_s - \int_t^T g\left(s, Y_s^m, Z_s^m\right) dB_s\right| \\ & + \left|\int_t^T Z_s^n dW_s - \int_t^T Z_s^m dW_s\right|, \end{aligned}$$

and taking limits on m and supremum over t, we get

$$\begin{aligned} \sup_{0\leq t\leq T} \left|Y_t^n - Y_t\right| & \leq \int_0^T \left|f_n\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s, Z_s\right)\right| ds \\ & + \sup_{0\leq t\leq T} \left|\int_t^T g\left(s, Y_s^n, Z_s^n\right) dB_s - \int_t^T g\left(s, Y_s, Z_s\right) dB_s\right| \\ & + \sup_{0\leq t\leq T} \left|\int_t^T Z_s^n dW_s - \int_t^T Z_s dW_s\right|, \quad \mathbb{P}-a.s. \end{aligned}$$

From which it follows that Y^n converges uniformly in t to Y in particular,Y is a continuous process. Not that $\{Y^n\}$ is monotone; therefore, we actually have the uniform convergence for the entire sequence and not just for a subsequence. Taking limits in Eq (2.9) we deduce that (Y, Z) is a solution of Eq(2.1) .

Let $\left(\hat{Y}, \hat{Z}\right) \in \mathcal{S}^2\left([0,T];\mathbb{R}\right) \times \mathcal{H}^2\left(0,T;\mathbb{R}\right)$ be any solution of Eq (2.1) .From theorem (15) , we get that $Y^n \leq \hat{Y}, \forall n \in \mathbb{N}$ and therefore $Y \leq \hat{Y}$ proving that Y is the minimal solution. ■

2.4 Reflected Backward doubly stochastic differential equations

In this section, we study the case where the solution is forced to stay above a given stochastic process, called the obstacle. We obtain the real valued reflected backward doubly stochastic differential equation :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, dB_s + K_T - K_t - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T \quad (2.12)$$

We consider the following conditions,

H1) Let $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable and such that for every $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $f(., y, z) \in \mathcal{H}^2(0, T, \mathbb{R})$, $g(., y, z) \in \mathcal{H}^2(0, T, \mathbb{R})$

H2) There exist constants $L > 0$ et $0 < \alpha < 1$, such that for every $(t, \omega) \in \Omega \times [0, T]$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{cases} |f(t, y, z) - f(t, y', z')| \leq L(|y - y'| + |z - z'|) \\ |g(t, y, z) - g(t, y', z')|^2 \leq L|y - y'|^2 + \alpha |z - z'|^2 \end{cases}$$

H3) Let ξ be a square integrable random variable which is $\mathcal{F}_T-$mesurable.

H4) The obstacle $\{S_t, 0 \leq t \leq T\}$, is a continuous $\mathcal{F}_t-$progressively measurable real-valued process satisfying

$$iv) E\left(\sup_{0 \leq t \leq T} (S_t)^2\right) < \infty.$$

We assume also that $S_T \leq \xi$ $a.s.$

Definition 18 *A solution of equation (2.12) is a* $(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+)$ *-valued* $\mathcal{F}_t-$*progressively measurable process* $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$ *which satisfies equations (2.12) and*

v) $(Y, Z, K_T) \in \mathcal{S}^2 \times \mathcal{H}^2 \times L^2(\Omega)$.

vi) $Y_t \geq S_t$.

vii) (K_t) *is continuous and nondecreasing,* $K_0 = 0$ *and* $\int_0^T (Y_t - S_t)\, dK_t = 0$.

2.4.1 Existence of a solution of the RBDSDE with lipschitz condition

Theorem 19 *Under conditions, H1), H2), H3) and H4), the RBDSDE (2.12) has unique solution.*

Remark. In the sequel C will be note a constant which may changes from line to line.

Lemma 20 *Let $i = 1, 2$. Let (η^i) be a square integrable and $\mathcal{G}_T$-measurable. Let $h^i : [0,T] \times \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ be such that for every $\mathcal{G}_t$-adapted process Y satisfying $E(sup_{t\leq T} {Y_t}^2) < \infty$, we have $h^i(s, Y_s)$ is $\mathcal{G}_t$-adapted and $\mathbb{E}\int_0^T (h^i(s, Y_s))^2 ds < \infty$. Let (Y^i, Z^i) be a solution of following BSDE :*

$$\begin{cases} Y_t^i = \eta^i + \int_t^T h^i(s, Y_s^i)ds - \int_t^T Z_s^i dW_s \\ \mathbb{E}(sup_{t\leq T}|Y_t^i|^2 + \int_0^T |Z_s^i|^2 ds) < \infty \end{cases}$$

Assume that,

i) h^1 is a uniformly lipischitz function in the variable y.

ii) $\eta^1 \leq \eta^2$ a.s.

iii) $h^1(t, Y_t^2) \leq h^2(t, Y_t^2)$ $dP \times dt$ a.e.

Then,

$$Y_t^1 \leq Y_t^2, \quad 0 \leq t \leq T \quad a.s.$$

Proof. Applying Itô's formula to $\left|(Y_t^1 - Y_t^2)^+\right|^2$ and using the fact that $\eta^1 \leq \eta^2$, we obtain

$$\begin{aligned}
&\left|\left(Y_t^1-Y_t^2\right)^+\right|^2+\int_t^T 1_{\{Y_s^1>Y_s^2\}}\left|Z_s^1-Z_s^2\right|^2 ds \\
&\leq 2\int_t^T \left(Y_s^1-Y_s^2\right)^+\left(h^1\left(s,Y_s^1\right)-h^2\left(s,Y_s^2\right)\right)ds \\
&\quad -2\int_t^T \left(Y_s^1-Y_s^2\right)^+\left(Z_s^1-Z_s^2\right)dW_s.
\end{aligned}$$

Using the fact that h^1 is Lipischitz and Gronwall's lemma, we get $(Y_t^1-Y_t^2)^+=0$, for all $0\leq t\leq T$ a.s. Which implies that $Y_t^1\leq Y_t^2$, $\forall t$, a.s. ∎

We first consider the following simple RBDSDE, with f,g independent from (Y,Z).

$$\begin{cases}
Y_t=\xi+\int_t^T f(s)\,ds+K_T-K_t+\int_t^T g(s)\,dB_s-\int_t^T Z_s dW_s \\
Y_t\geq S_t \\
\int_0^T (Y_s-S_s)\,dK_s=0
\end{cases} \tag{2.13}$$

Proposition 21 *There exists a unique process (Y,Z,K) which solves equation (2.13).*

Proof. By [43], for $n\in\mathbb{N}$, let $(Y_t^n,Z_t^n)_{0\leq t\leq T}$ denote the unique pair , with values in $\mathbb{R}\times\mathbb{R}^d$ satisfying, $(Y^n,Z^n)\in\mathcal{S}^2\times\mathcal{H}^2$ and

$$Y_t^n:=\xi+\int_t^T f(s)\,ds+n\int_t^T (S_s-Y_s^n)^+ds+\int_t^T g(s)\,dB_s-\int_t^T Z_s^n dW_s.$$

We define

$$\begin{cases}
\overline{\xi}:=\xi+\int_0^T f(s)\,ds+\int_0^T g(s)\,dB_s \\
\overline{S}_t:=S_t+\int_0^t f(s)\,ds+\int_0^t g(s)\,dB_s \\
\overline{Y}_t^n:=Y_t^n+\int_0^t f(s)\,ds+\int_0^t g(s)\,dB_s
\end{cases}$$

we have,

$$\overline{Y}_t^n=\overline{\xi}+n\int_t^T \left(\overline{S_s}-\overline{Y}_s^n\right)^+ds-\int_t^T Z_s^n dW_s. \tag{2.14}$$

Let $\Lambda_t = \mathbb{E}^{\mathcal{G}_t}\left[\overline{\xi} \vee sup_{s\leq T}\overline{S}_s\right]$. Then there exists $\gamma \in \mathbb{L}^2([0,T]\times\Omega, \mathbb{R}^d)$ which is $\mathcal{G}_t$-predictable such that

$$\Lambda_t = \Lambda_T - \int_t^T \gamma_s dW_s \tag{2.15}$$

Since $(\overline{S}_s - \Lambda_s)^+ = 0$, we have

$$\Lambda_t = \Lambda_T + n\int_t^T (\overline{S}_s - \Lambda_s)^+ ds - \int_t^T \gamma_s dW_s \tag{2.16}$$

By Lemma (20) we have

$$\overline{Y}_t^0 = \mathbb{E}^{\mathcal{G}_t}\left[\overline{\xi}\right] \leq \overline{Y}_t^n \leq \overline{Y}_t^{n+1} \leq \Lambda_t = \mathbb{E}^{\mathcal{G}_t}\left[\overline{\xi} \vee sup_{s\leq T}\overline{S}_s\right].$$

Set $\overline{Y}_t := \sup_n \overline{Y}_t^n$ and $Y_t := \sup_n Y_t^n$.

Applying Itô's formula, and passing to expectation we get

$$\mathbb{E}\int_0^T |\gamma_s - Z_s^n|^2 ds \leq \mathbb{E}|\sup_{s\leq T}(\overline{S}_s - \overline{\xi})^+|^2$$

Hence

$$\mathbb{E}\int_0^T |Z_s^n|^2 ds \leq 2E|\sup_{s\leq T}(\overline{S}_s - \overline{\xi})^+|^2 + 2\mathbb{E}\int_0^T |\gamma_s|^2 ds$$

Coming back to equation (2.14) and using equation (2.15) we obtain

$$\mathbb{E}(n\int_0^T (S_s - Y_s^n)^+ \, ds)^2 \leq 4\mathbb{E}|\sup_{s\leq T}(\overline{S}_s - \overline{\xi})^+|^2$$

Hence, there exist a nondecreasing and right continuous process K satisfying $\mathbb{E}(K_T^2) < \infty$ such that for a subsequence of n (which still denoted n) we have for all $\varphi \in \mathbb{L}^2(\Omega; \mathcal{C}([0,T]))$,

$$\lim_n \mathbb{E}\int_0^T \varphi_s n(S_s - Y_s^n)^+ ds = \mathbb{E}\int_0^T \varphi_s dK_s.$$

Let $N \in \mathbb{N}^*$ and $n, m \geq N$. We have

$$(Y_t^n - Y_t^m)^2 \leq 2\int_t^T (S_s - Y_s^N) n (S_s - Y_s^n)^+ ds + 2\int_t^T (S_s - Y_s^N) m (S_s - Y_s^m)^+ ds$$
$$- 2\int_t^T (Z_s^n - Z_s^m)(Y_s^n - Y_s^m) dW_s - \int_t^T |Z_s^n - Z_s^m|^2 ds$$

By BDG inequality, there exists a constant C such that

$$\limsup_{n,m} \left(\mathbb{E}\left(\sup_{t \leq T} (Y_t^n - Y_t^m)^2 \right) + \mathbb{E}\int_0^T |Z_s^n - Z_s^m|^2 \, ds \right) \leq 2C\mathbb{E}\int_0^T \left(S_s - Y_s^N\right) dK_s$$

Letting N tends to ∞, we obtain

$$\limsup_{n,m} \left(\mathbb{E}\left(\sup_{t \leq T} (Y_t^n - Y_t^m)^2 \right) + \mathbb{E}\int_0^T |Z_s^n - Z_s^m|^2 \, ds \right) \leq 2C\mathbb{E}\int_0^T (S_s - Y_s) \, dK_s$$

Let

$$\widetilde{Y}_t^n := \overline{S}_T + n\int_t^T \left(\overline{S}_s - \widetilde{Y}_s^n\right) ds - \int_t^T \widetilde{Z}_s^n dW_s.$$

Since $\overline{S}_T \leq \overline{\xi}$, the comparison theorem, shows that, for every n we have, $\forall t \in [0, T]$, $\overline{Y}_t^n \geq \widetilde{Y}_t^n$ $a.s.$ Let σ be a $\mathcal{G}_t-$stopping time, and $\tau = \sigma \wedge T$. We have

$$\widetilde{Y}_\tau^n = \mathbb{E}^{\mathcal{G}_\tau}\left[\overline{S}_T e^{-n(T-\tau)} + n\int_\tau^T \overline{S}_s e^{-n(s-\tau)} ds\right]$$

It is not difficult to see that $\widetilde{Y}^n$ converges to $\overline{S}_\tau$ $a.s.$ Therefore $\overline{Y}_\tau \geq \overline{S}_\tau$ $a.s.$, and hence $Y_\tau \geq S_\tau$ $a.s.$

Using section theorem, we get, $a.s.$ for every $t \in [0, T]$, $Y_t \geq S_t$, which implies that

$$\limsup_{n,m} (\mathbb{E}(\sup_{t \leq T} (Y_t^n - Y_t^m)^2) + \mathbb{E}\int_0^T |Z_s^n - Z_s^m|^2 ds) = 0$$
$$\text{and } \mathbb{E}\int_0^T (S_s - Y_s) dK_s = 0$$

We deduce that (Y, K) is continuous and there exists Z in $\mathbb{L}^2$ such that Z^n converges strongly in $\mathbb{L}^2$ to Z. Finally, it is not difficult to check that (Y, Z, K) satisfies equation (2.13) ■

Proof. of Theorem19 Existence. We define a sequence $(Y_t^n, Z_t^n, K_t^n)_{0\leq t\leq T}$ as follows. Let $Y_t^0 = S_t,\ Z_t^0 = 0$ and for $t \in [0,T]$ and $n \in \mathbb{N}^*$,

$$\begin{cases} Y_t^{n+1} = \xi + \int_t^T f\left(s, Y_s^n, Z_s^n\right) ds + \int_t^T g\left(s, Y_s^n, Z_s^n\right) dB_s + \int_t^T dK_s^{n+1} - \int_t^T Z_s^{n+1} dW_s \\ Y_t^{n+1} \geq S_t \quad \text{a.s.} \\ \left(Y_s^{n+1} - S_s\right) dK_s^{n+1} = 0 \end{cases}$$

Such sequence $\left(Y^n, Z^n, K^n\right)_n$ exists by previous step.

Put $\overline{Y}^{n+1} = Y^{n+1} - Y^n$. By Itô's formula, we have,

$$\left|\overline{Y}_t^{n+1}\right|^2 + \int_t^T \left|\overline{Z}_s^{n+1}\right|^2 ds = 2\int_t^T \overline{Y}_s^{n+1}\left(f\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) ds$$
$$+ \int_t^T \overline{Y}_s^{n+1}\left(dK_s^{n+1} - dK_s^n\right) + 2\int_t^T \overline{Y}_s^{n+1}\left(g\left(s, Y_s^n, Z_s^n\right) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) dB_s$$
$$+2\int_t^T \overline{Y}_s^{n+1}\overline{Z}_s^{n+1} dW_s + \int_t^T \left|g\left(s, Y_s^n, Z_s^n\right) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right|^2 ds$$

Therefore, Itô's formula applied to $|y|^2 e^{\beta t}$ shows that :

$$\begin{aligned} &\left|\overline{Y}_t^{n+1}\right|^2 e^{\beta t} - \beta\int_t^T \left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s} ds + \int_t^T e^{\beta s}\left|\overline{Z}_s^{n+1}\right|^2 ds \\ &= 2\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\left(f\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) ds \\ &+ \int_t^T e^{\beta s}\overline{Y}_s^{n+1}\left(dK_s^{n+1} - dK_s^n\right) \\ &+2\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\left(g\left(s, Y_s^n, Z_s^n\right) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) dB_s \\ &+2\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\overline{Z}_s^{n+1} dW_s \\ &+ \int_t^T e^{\beta s}\left|g\left(s, Y_s^n, Z_s^n\right) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right|^2 ds \end{aligned}$$

Shows that,

$$\mathbb{E}\left(\left|\overline{Y}_t^{n+1}\right|^2\right) e^{\beta t} - (\beta + 2C\gamma)\,\mathbb{E}\left(\int_t^T \left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s}\right) ds + \mathbb{E}\int_t^T \left|\overline{Z}_s^{n+1}\right|^2 ds$$
$$\leq \left(C + \frac{2C}{\gamma}\right)\mathbb{E}\int_t^T \left|\overline{Y}_s^n\right|^2 e^{\beta s} ds + \left(\alpha + \frac{2C}{\gamma}\right)\mathbb{E}\int_t^T \left|\overline{Z}_s^n\right|^2 e^{\beta s} ds$$

Choosing $\gamma = \frac{4C}{(1-\alpha)}$, $\overline{C} = \frac{2}{1+\alpha}\left(C + \frac{1-\alpha}{2}\right)$, and $\beta = -2C\gamma - \overline{C}$, we have

$$\mathbb{E}\int_t^T \left(\overline{C}\left|\overline{Y}_s^{n+1}\right|^2 + \left|\overline{Z}_s^{n+1}\right|^2\right) e^{\beta s} ds$$
$$\leq \left(\frac{1+\alpha}{2}\right)^n \mathbb{E}\int_t^T \left(\overline{C}\left|\overline{Y}_s^1\right|^2 + \left|\overline{Z}_s^1\right|^2\right) e^{\beta s} ds$$

Since $\frac{1+\alpha}{2} < 1$, there exists (Y, Z) in $\mathcal{H}^2 \times \mathcal{H}^2$ such that (Y^n, Z^n) converges to (Y, Z) in $\mathcal{H}^2 \times \mathcal{H}^2$. It is not difficult to deduce that Y^n converges to Y in S^2.

It is not difficult to shows that (Y, Z, K) is a solution to RBDSDE (2.12).

Uniqueness. It follows the comparison theorem which will be established below. ■

2.4.2 RBDSDEs with continuous coefficient

In this section we prove the existence of solution to RBDSDE when the coefficient is only continuous.

We consider the following assumption

H5) i) for $a.e$ (t, w), the map $(y, z) \mapsto f(t, y, z)$ is continuous.

ii) There exist constants $\kappa > 0$, $L > 0$ and $\alpha \in]0, 1[$, such that for every $(t, \omega) \in \Omega \times [0, T]$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{cases} |f(t, y, z)| \leq \kappa(1 + |y| + |z|) \\ |g(t, y, z) - g(t, y', z')|^2 \leq L|y - y'|^2 + \alpha|z - z'|^2 \end{cases}$$

Theorem 22 *Under assumption H1), H3), H4) and H5), the RBDSDE (2.12) has an adapted solution (Y, Z, K) which is a minimal one, in the sense that, if (Y^*, Z^*) is any other solution we have $Y \leq Y^*$, $P - a.s.$*

Before giving the proof of Theorem 22, we recall the following classical lemma. It can be proved by adapting the proof given in J. J. Alibert and K. Bahlali [1].

Lemma 23 *Let $f : [0,T] \times \Omega \times \mathbb{R}^d \longmapsto \mathbb{R}$ be a measurable function such that:*

(a) For almost every $(t,\omega) \in [0,T] \times \Omega$, $x \longmapsto f(t,x)$ is continuous,

(b) There exists a constant $K > 0$ such that for every $(t,x) \in [0,T] \times \mathbb{R}^d$ $|f(t,x)| \leq K(1+|x|)$ a.s.

Then, the sequence of functions

$$f_n(t,x) = \inf_{y\in\mathbb{Q}^d} \{h(t,y) + n|x-y|\}$$

is well defined for each $n \geq K$ and satisfies:

(1) for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $|f_n(t,x)| \leq K(1+|x|)$,

(2) for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $n \to f_n(t,x)$ is increasing,

(3) for every $n \geq K$, $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$, $|f_n(t,x) - f_n(t,y)| \leq n|x-y|$,

(4) If $x_n \to x$, as $n \to \infty$ then for every $t \in [0,T]$ $f_n(t,x_n) \to f(t,x)$ as $n \to \infty$.

Since ξ satisfies H3), we get from Theorem (19), that for every $n \in \mathbb{N}^*$, there exists a unique solution $\{(Y_t^n, Z_t^n, K_t^n),\ 0 \leq t \leq T\}$ for the following RBDSDE

$$\begin{cases} Y_t^n = \xi + \int_t^T f_n(s,Y_s^n,Z_s^n)ds + K_T^n - K_t^n + \int_t^T g(s,Y_s^n,Z_s^n)dB_s - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T, \\ Y_t^n \geq S_t \\ \int_0^T (Y_s^n - S_s)\, dK_s^n = 0 \end{cases} \tag{2.17}$$

We consider the function defined by

$$f^1(t,u,v) := \kappa(1+|u|+\|v\|)$$

Since, $|f^1(t,u,v)-f^1(t,u',v')| \leq \kappa(|u-u'|+\|v-v'\|)$, then similar argument as before shows that there exists a unique solution $((U_s,V_s,K_s), 0 \leq s \leq T)$ to the following RBDSDE:

$$\begin{cases} U_t = \xi + \int_t^T f^1(s,U_s,V_s)ds + K_T - K_t + \int_t^T g(s,U_s,V_s)dB_s - \int_t^T V_s dW_s \\ U_t \geq S_t \\ \int_0^T (U_s - S_s)\, dK_s = 0 \end{cases} \tag{2.18}$$

We need also the following comparison theorem

Theorem 24 *Let (ξ, f, g, S) and (ξ', f', g, S') be two RBDSDEs. Each one satisfying all the previous assumptions H1), H2), H3) and H4). Assume moreover that :*

i) $\xi \leq \xi'$ a.s.

ii) $f(t,y,z) \leq f'(t,y,z)$ $d\mathbb{P}\times dt$ a.e. $\forall (y,z) \in \mathbb{R}\times\mathbb{R}^d$.

iii) $S_t \leq S'_t$, $0 \leq t \leq T$ a.s.

Let (Y,Z,K) be a solution of RBDSDE (ξ, f, g, S) and (Y', Z', K') be a solution of RBDSDE (ξ', f', g, S'). Then

$$Y_t \leq Y'_t, \quad 0 \leq t \leq T \quad a.s.$$

Proof. Applying Itô's formula to $\left|(Y_t - Y'_t)^+\right|^2$, and passing to expectation, we have

$$\begin{aligned} &\mathbb{E}\left|(Y_t - Y'_t)^+\right|^2 + \mathbb{E}\int_t^T 1_{\{Y_s > Y'_s\}}\left|Z_s - Z'_s\right|^2 ds \\ &= 2\mathbb{E}\int_t^T \left(Y_s - Y'_s\right)^+ \left(f(s,Y_s,Z_s) - f'\left(s,Y'_s,Z'_s\right)\right) ds \\ &+ 2\mathbb{E}\int_t^T \left(Y_s - Y'_s\right)^+ \left(dK_s - dK'_s\right) \\ &+ \mathbb{E}\int_t^T \left|g(s,Y_s,Z_s) - g\left(s,Y'_s,Z'_s\right)\right|^2 1_{\{Y_s > Y'_s\}} ds. \end{aligned}$$

Since on the set $\{Y_s > Y'_s\}$, we have $Y_t > S'_t \geq S_t$, then

$$\int_t^T \left(Y_s - Y_s^{'}\right)^+ \left(dK_s - dK'_s\right) = -\int_t^T \left(Y_s - Y_s^{'}\right)^+ dK'_s \leq 0$$

Since f is Lipschitz, we have on the set $\{Y_s > Y'_s\}$,

$$\begin{aligned}
&\mathbb{E}\left|(Y_t - Y'_t)^+\right|^2 + \mathbb{E}\int_t^T 1_{\{Y_s > Y'_s\}} \left|Z_s - Z_s^{'}\right|^2 ds \\
&\leq \left(3C + \frac{1}{\varepsilon}C^2\right) \mathbb{E}\int_t^T \left|Y_s - Y_s^{'}\right|^2 1_{\{Y_s > Y'_s\}} ds \\
&+ (\varepsilon + \alpha)\, \mathbb{E}\int_t^T \left|Z_s - Z_s^{'}\right|^2 1_{\{Y_s > Y'_s\}} ds.
\end{aligned}$$

We now choose $\varepsilon = \frac{1-\alpha}{2}$, and $\overline{C} = 3C + \frac{1}{\varepsilon}C^2$, to deduce that

$$\mathbb{E}\left|(Y_t - Y'_t)^+\right|^2 \leq \overline{C}\mathbb{E}\int_t^T \left|(Y_s - Y_s^{'})^+\right|^2 ds.$$

The result follows now by using Gronwall's lemma. ■

Lemma 25 *i) a.s. for all, t* $Y_t^0 \leq Y_t^n \leq Y_t^{n+1} \leq U_t$.

ii) There exists $Z \in \mathcal{H}^2$ *such that* Z^n *converges to* Z.

Proof. Assertion i) follows from Theorem (24). We shall prove ii).

Itô's formula yields

$$\begin{aligned}
\mathbb{E}|Y_0^n|^2 + \mathbb{E}\int_0^T \|Z_s^n\|^2 ds =& \mathbb{E}|\xi|^2 + 2E\int_0^T Y_s^n f_n(s, Y_s^n, Z_s^n) ds + 2\mathbb{E}\int_0^T S_s dK_s^n \\
&+ \mathbb{E}\int_0^T \|g(s, Y_s^n, Z_s^n)\|^2 ds
\end{aligned}$$

But, assumption H5) and the inequality $2ab \leq \dfrac{a^2}{r} + rb^2$ for $r > 0$, show that :

$$\begin{aligned}
\mathbb{E}\int_0^T \|Z_s^n\|^2 ds &\leq C + \left(r\kappa^2 + (1+\varepsilon)\alpha\right) \mathbb{E}\int_0^T \|Z_s^n\|^2 ds + 2E\int_0^T S_s dK_s^n \\
&\leq C + \left(r\kappa^2 + (1+\varepsilon)\alpha\right) \mathbb{E}\int_0^T \|Z_s^n\|^2 ds + \beta\mathbb{E}(K_T^n)^2
\end{aligned}$$

On the other hand, we have from (2.17)

$$K_T^n = Y_0^n - \xi - \int_0^T f_n(s, Y_s^n, Z_s^n)ds - \int_0^T g(s, Y_s^n, Z_s^n)dB_s + \int_0^T Z_s^n dW_s \tag{2.19}$$

then

$$\mathbb{E}(K_T^n)^2 \leq C\left(1 + \mathbb{E}\int_0^T \|Z_s^n\|^2 ds\right)$$

which yield that

$$\mathbb{E}\int_0^T \|Z_s^n\|^2 ds \leq C + \left(r\kappa^2 + (1+\varepsilon)\alpha + \beta C\right)\mathbb{E}\int_0^T \|Z_s^n\|^2 ds$$

Choosing $r = \varepsilon = \beta = \frac{1-\alpha}{2(\kappa^2+\alpha+C)}$, we obtain

$$\mathbb{E}\int_0^T \|Z_s^n\|^2 ds \leq C$$

For $n, p \geq K$, Itô's formula gives,

$$\begin{aligned}\mathbb{E}(Y_0^n - Y_0^p)^2 + \mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds =& 2\mathbb{E}\int_0^T (Y_s^n - Y_s^p)(f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))ds \\ &+ 2\mathbb{E}\int_0^T (Y_s^n - Y_s^p)dK_s^n + 2\mathbb{E}\int_0^T (Y_s^p - Y_s^n)dK_s^p \\ &+ \mathbb{E}\int_0^T \|g(s, Y_s^n, Z_s^n) - g(s, Y_s^p, Z_s^p)\|^2 ds.\end{aligned}$$

But

$$\mathbb{E}\int_0^T (Y_s^n - Y_s^p)dK_s^n = \mathbb{E}\int_0^T (S_s - Y_s^p)dK_s^n \leq 0$$

Similarly, we have $\mathbb{E}\int_0^T (Y_s^p - Y_s^n)dK_s^p \leq 0$.

Therefore,

$$\begin{aligned}\mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds \leq& 2\mathbb{E}\int_0^T (Y_s^n - Y_s^p)(f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))ds \\ &+ \mathbb{E}\int_0^T \|g(s, Y_s^n, Z_s^n) - g(s, Y_s^p, Z_s^p)\|^2 ds\end{aligned}$$

By Hôlder's inequality and the fact that g is Lipschitz, we get

$$
\begin{aligned}
&\mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds \\
&\leq \left(\mathbb{E}\int_0^T (Y_s^n - Y_s^p)^2 ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T (f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p))^2 ds\right)^{\frac{1}{2}} \\
&+ C\mathbb{E}\int_0^T |Y_s^n - Y_s^p|^2 ds + \alpha\mathbb{E}\int_0^T |Z_s^n - Z_s^p|^2 ds
\end{aligned}
$$

Since $\sup_n \mathbb{E}\int_0^T |f_n(s, Y_s^n, Z_s^n)|^2 \leq C$, we obtain,

$$\mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds \leq C\left(\mathbb{E}\int_0^T (Y_s^n - Y_s^p)^2 ds\right)^{\frac{1}{2}}$$

Hence

$$\mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds \longrightarrow 0;\ as \ \ n, p \to \infty$$

Thus $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}^2(\mathbb{R}^d)$. ■

Proof. of theorem 22. Put $Y_t = sup_n Y_t^n$, we have $(Y^n, Z^n) \to (Y, Z)$ in $S^2(\mathbb{R}^d) \times \mathcal{H}^2(\mathbb{R}^d)$. Then, along a subsequence which we still denote (Y^n, Z^n), we get

$$(Y^n, Z^n) \to (Y, Z), \quad dt \otimes d\mathbb{P}\ a.e$$

then, using Lemma 23, we get $f_n(t, Y_t^n, Z_t^n) \to f(t, Y_t, Z_t) \quad d\mathbb{P} \otimes dt\ a.e$

On the other hand, since $Z^n \longrightarrow Z$ in $\mathcal{H}^2(\mathbb{R}^d)$, then there exists $\Lambda \in \mathcal{H}^2(\mathbb{R})$ and a subsequence which we still denote Z^n such that $\forall n, |Z^n| \leq \Lambda,\ Z^n \longrightarrow Z,\ dt \otimes d\mathbb{P}\ a.e.$

Moreover from H5), and Lemma 31 we have

$$|f_n(t, Y_t^n, Z_t^n)| \leq \kappa(1 + \sup_n |Y_t^n| + \Lambda_t) \in \mathbb{L}^2([0, T],\ dt), \quad \mathbb{P} - a.s,$$

It follows from the dominated convergence theorem that,

$$\mathbb{E}\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^2\, ds \longrightarrow 0, \quad n \to \infty. \tag{2.20}$$

We have,

$$\mathbb{E}\int_0^T \|g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)\|^2 ds$$
$$\leq C\mathbb{E}\int_0^T |Y_s^n-Y_s|^2 ds+\alpha\mathbb{E}\int_0^T \|Z_s^n-Z_s\|^2 ds \longrightarrow 0, \quad \text{as } n\to\infty.$$

It is not difficult to show that (Y,Z) is solution to our RBDSDE. Let

$$\overline{Y}_t=\xi+\int_t^T f\left(s,Y_s,Z_s\right)ds+K_T-K_t+\int_t^T g\left(s,Y_s,Z_s\right)dB_s-\int_t^T \overline{Z}_s dW_s, \qquad (2.21)$$

$\overline{Z}\in\mathcal{H}^2$, $\overline{Y}\in S^2$, $K_T\in\mathbb{L}^2$, $\overline{Y}_t\geq S_t$, (K_t) is continuous and nondecreasing, $K_0=0$ and $\int_0^T\left(\overline{Y}_t-S_t\right)dK_t=0$, and (Y^*,Z^*,K^*) be a solution of (2.12). Then, by Theorem (24), we have for every $n\in\mathbb{N}^*$, $Y^n\leq Y^*$. therefore, $\overline{Y}$ is a minimal solution of (2.12) ■

Chapter 3

Mean-Field Reflected Backward Doubly Stochastic Differential Equations

In this chapter we study the existence and uniqueness of the solutions to mean-field reflected backward doubly stochastic differential equation when the driver f is lipschitz. We also study the existence in the case when the driver is of linear growth and continuous, in this case we establish a comparison theorem.

3.1 MF-RBDSDE with lipschitz condition

We shall consider the following $\mathbf{MF-RBDSDE}$:

$$\begin{aligned} Y_t &= \xi+\int_t^T \mathbb{E}' f\left(s,\omega,\omega',Y_s,Y_s',Z_s,Z_s'\right) ds+\int_t^T \mathbb{E}' g\left(s,\omega,\omega',Y_s,Y_s',Z_s,Z_s'\right) dB_s \quad (3.1)\\ & +K_T-K_t-\int_t^T Z_s dW_s. \end{aligned}$$

3.1.1 Assumptions and Definitions

We consider the following assumptions,

H1) Let $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\rightarrow\mathbb{R}$ and $g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\rightarrow\mathbb{R}$ be two measurable functions and such that for every $(y,z,y',z')\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$, $f(.,y,z,y',z')$ and, $g(.,y,z,y',z')$ belongs in $\mathcal{H}^2(0,T,\mathbb{R})$

H2) There exist constants $L>0$ and $0<\alpha<\frac{1}{2}$, such that for every $(t,\omega)\in\Omega\times[0,T]$ and $(y,z,y',z')\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$,

$$\left|f\left(t,y_1,z_1,y_1',z_1'\right)-f\left(t,y_2,z_2,y_2',z_2'\right)\right|$$

$$\leq L\left(|y_1-y_2|+\left|y_1'-y_2'\right|+|z_1-z_2|+\left|z_2'-z_2'\right|\right)$$

$$\left|g\left(t,y_1,z_1,y_1',z_1'\right)-g\left(t,y_2,z_2,y_2',z_2'\right)\right|^2\leq$$

$$L\left(|y_1-y_2|^2+\left|y_1'-y_2'\right|^2\right)+\alpha\left(|z_1-z_2|^2+\left|z_1'-z_2'\right|^2\right)$$

H3) Let ξ be a square integrable random variable which is $\mathcal{F}_T-$mesurable.

H4) The obstacle $\{S_t,0\leq t\leq T\}$, is a continuous $\mathcal{F}_t-$progressively measurable real-valued process satisfying $E\left(\sup_{0\leq t\leq T}(S_t)^2\right)<\infty$.

We assume also that $S_T\leq\xi$ $a.s.$

Definition 26 *A solution of equation (3.1) is a* $\left(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}_+\right)$ *-valued* $\mathcal{F}_t-$*progressively measurable process* $(Y_t,Z_t,K_t)_{0\leq t\leq T}$ *which satisfies equation (3.1) and*

i) $(Y,Z,K_T)\in S^2\times\mathcal{H}^2\times L^2(\Omega)$.

ii) $Y_t\geq S_t$.

iii) (K_t) *is continuous and nondecreasing,* $K_0=0$ *and* $\int_0^T(Y_t-S_t)\,dK_t=0$.

3.1.2 Existence of a solution to the MF-RBDSDE with lipschitz condition

Theorem 27 *Under conditions, H1), H2), H3) and H4), the MF-RBDSDE:*(3.1)*has a unique solution.*

Proof. For any (y,z) we consider the following MF-RBDSDE, with $t\in[0,T]$

$$\begin{aligned}Y_t &= \xi+\int_t^T\mathbb{E}'f\left(s,\omega,\omega',Y_s,y'_s,Z_s,z'_s\right)ds+\int_t^T\mathbb{E}'g\left(s,\omega,\omega',Y_s,y'_s,Z_s,z'_s\right)dB_s\\ &\quad +K_T-K_t-\int_t^T Z_s dW_s.\end{aligned}$$

According to Theorem 19, there exists a unique solution $(Y,Z)\in S^2\times\mathcal{H}^2$ i.e., if we define the process

$$\begin{aligned}K_t &= Y_0-Y_t-\int_0^t\mathbb{E}'f\left(s,\omega,\omega',Y_s,y'_s,Z_s,z'_s\right)ds-\int_0^t\mathbb{E}'g\left(s,\omega,\omega',Y_s,y'_s,Z_s,z'_s\right)dB_s\\ &\quad +\int_0^t Z_s dW_s,\end{aligned}$$

then (Y,Z,K) satisfies Definition 26. Hence, if we define $\Theta(y,z)=(Y,Z)$, then Θ maps $S^2\times\mathcal{H}^2$ itself. We show now that Θ is contractive. To this end, take any $\left(y^i,z^i\right)\in S^2\times\mathcal{H}^2$ $(i=1,2)$, and let $\Theta(y^i,z^i)=(Y^i,Z^i)$. We denote $\left(\overline{Y},\overline{Z},\overline{K}\right)=\left(Y^1-Y^2,Z^1-Z^2,K^1-K^2\right)$ and $(\overline{y},\overline{z})=\left(y^1-y^2,z^1-z^2\right)$. Therefore, Itô's formula applied to $\left|\overline{Y}\right|^2e^{\beta t}$ where $\beta>0$, and the inequality $2ab \leq \left(\frac{1}{\delta}\right) a^2 + \delta b^2$, lead to

$$\begin{array}{l}\mathbb{E}\left|\overline{Y}_t\right|^2 e^{\beta t} + (\beta - 3L - \frac{8L^2}{1-2\alpha})\mathbb{E}\int_t^T \left|\overline{Y}_s\right|^2 e^{\beta s} ds + \frac{1}{2}\mathbb{E}\int_t^T e^{\beta s}\left|\overline{Z}_s\right|^2 ds \\ \leq \mathbb{E}\int_t^T e^{\beta s}\overline{Y}_s\left(dK_s^1 - dK_s^2\right) \\ +\mathbb{E}\int_t^T e^{\beta s}\left((L + \frac{1-2\alpha}{2L})\left|\overline{y}_s\right|^2 + (\frac{1+2\alpha}{4})\left(\left|\overline{z}_s\right|^2\right)\right) ds\end{array}$$

Choosing $\beta = 3L + \frac{8L^2}{1-2\alpha} + \frac{1}{2}\left(\frac{4}{1+2\alpha}\right)\left(L + \frac{1-2\alpha}{2L}\right)$ and setting $M = \left(\frac{4}{1+2\alpha}\right)\left(L + \frac{1-2\alpha}{2L}\right)$ yield

$$\begin{array}{l}\mathbb{E}\left|\overline{Y}_t\right|^2 e^{\beta t} + \frac{1}{2}M\mathbb{E}\int_t^T \left|\overline{Y}_s\right|^2 e^{\beta s} ds + \frac{1}{2}\mathbb{E}\int_t^T e^{\beta s}\left|\overline{Z}_s\right|^2 ds \\ \leq \mathbb{E}\int_t^T e^{\beta s}\overline{Y}_s\left(dK_s^1 - dK_s^2\right) \\ +\frac{1+2\alpha}{4}\mathbb{E}\int_t^T e^{\beta s}\left(M\left|\overline{y}_s\right|^2 + \left|\overline{z}_s\right|^2\right) ds\end{array}$$

We have

$$\mathbb{E}\int_t^T e^{\beta s}\overline{Y}_s\left(dK_s^1 - dK_s^2\right) < 0,$$

and

$$\mathbb{E}\int_t^T e^{\beta s}\left(M\left|\overline{Y}_s\right|^2 + \left|\overline{Z}_s\right|^2\right) ds \leq \frac{1+2\alpha}{2}\mathbb{E}\int_t^T e^{\beta s}\left(M\left|\overline{y}_s\right|^2 + \left|\overline{z}_s\right|^2\right) ds$$

Consequently the mapping Θ is a strict contraction on $S^2 \times \mathcal{H}^2$ equipped with the norm

$$\|(Y,Z)\|_\beta = \left(\mathbb{E}\int_t^T e^{\beta s}\left(M\left|\overline{Y}_s\right|^2 + \left|\overline{Z}_s\right|^2\right) ds\right)^{\frac{1}{2}}.$$

Moreover, it has a unique fixed point, which is the unique solution of the MF-RBDSDE with data (ξ, f, g, S). ■

3.2 MF-RBDSDEs with continuous coefficient

In this section we prove the existence of a solution to the MF-RBDSDE where the coefficient is only continuous.

We consider the following assumption

H5) i) for $a.e$ (t,w), the mapping $(y,y',z,z') \mapsto f(t,y,y',z,z')$ is continuous.

ii) There exist constants $L>0$ and $\alpha \in]0,\frac{1}{2}[$, such that for every $(t,\omega) \in \Omega \times [0,T]$ and $(y,z,y',z') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{cases} |f(t,y,y',z,z')| \leq L(1+|y|+|y'|+|z|+|z'|) \\ |g(t,y_1,y_1',z_1,z_1') - g(t,y_2,y_2',z_2,z_2')|^2 \leq L\left(|y_1'-y_2'|^2+|y_1-y_2|^2\right) \\ \qquad +\alpha\left(|z_1'-z_2'|^2+|z_1-z_2|^2\right) \end{cases}$$

Theorem 28 *Under assumption H1), H3), H4) and H5), the MF-RBDSDE (3.1) has an adapted solution (Y,Z,K) which is a minimal one, in the sense that, if (Y^*,Z^*) is any other solution we have $Y \leq Y^*$, $P-a.s.$*

Before giving the proof of Theorem 28, we recall the following classical lemma. It can be proved by adapting the proof given in Alibert and Bahlali [1].

Lemma 29 *Let $f:[0,T]\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d \longmapsto \mathbb{R}$ be a measurable function such that:*

(a) For almost every $(t,\overline{\omega}) \in [0,T]\times\overline{\Omega}$, $x \longmapsto f(t,\overline{\omega},x)$ is continuous,

b) There exists a constant $K>0$ such that for every $(t,y',y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$

$$|f(t,y',y,z)| \leq K(1+|y'|+|y|+|z|) \ a.s.$$

(c) For almost every $y,z,f(t,y',y,z)$ is increasing in y'.

Then, the sequence of functions

$$f_n(t,y',y,z)=\inf_{(u,v,w)\in\mathbb{Q}^{2+d}}\Big\{f(t,u,v,w)+n\left(y'-u\right)^{+}+n|y-v|+n\left|z-w\right|\Big\}$$

is well defined for each $n\geq K$ and satisfies:

(1) for every $(t,y',y,z)\in[0,T]\times\mathbb{R}^{2+d}$, $|f_n(t,y',y,z)|\leq K(1+|y'|+|y|+|z|)$,

(2) for every $(t,y',y,z)\in[0,T]\times\mathbb{R}^{2+d}$, $n\rightarrow f_n(t,x)$ is increasing,

(3) for every $(t,y',y,z)\in[0,T]\times\mathbb{R}^{2+d}$, $y'\rightarrow f_n(t,y',y,z)$ is increasing,

(4) for every $n\geq K$, $(t,y'^1,y^1,z^1)\in[0,T]\times\mathbb{R}^{2d}$, $(t,y'^2,y^2,z^2)\in[0,T]\times\mathbb{R}^{2d}$

$$|f_n(t,y'^1,y^1,z^1)-f_n(t,y'^2,y^2,z^2)|\leq n\left(|y'^1-y'^2|+|y^1-y^2|+|z'^1-z'^2|\right),$$

(5) If $(y'_n,y_n,z_n)\rightarrow(y',y,z)$, as $n\rightarrow\infty$ then for every $t\in[0,T]$ $f_n(t,y'_n,y_n,z_n)\rightarrow f(t,y',y,z)$ as $n\rightarrow\infty$.

Since ξ satisfies H3), we get from Theorem (27), that for every $n\in\mathbb{N}^*$, there exists a unique solution $\{(Y_t^n,Z_t^n,K_t^n),\ 0\leq t\leq T\}$ for the following MF-RBDSDE

$$\begin{cases} Y_t^n=\xi+\int_t^T f_n(s,(Y_s^n)',Y_s^n,Z_s^n)ds+K_T^n-K_t^n+\int_t^T g(s,(Y_s^n)',Y_s^n,Z_s^n)dB_s \\ \qquad -\int_t^T Z_s^n dW_s, \quad 0\leq t\leq T, \\ Y_t^n\geq S_t \\ \int_0^T\left(Y_s^n-S_s\right)dK_s^n=0 \end{cases} \tag{3.2}$$

We consider the function defined by

$$f^1(t,u,v,w):=L(1+|u|+|v|+|w|),$$

since, $|f^1(t,u,v,w)-f^1(t,u',v',w')| \leq L(|u-u'|+|v-v'|+|w-w'|)$, then similar argument as before shows that there exists a unique solution $((U_s,V_s,K_s), 0 \leq s \leq T)$ to the following MF-RBDSDE:

$$\begin{cases} U_t = \xi + \int_t^T f^1(s,U'_s,U_s,V_s)ds + K_T - K_t + \int_t^T g(s,U'_s,U_s,V_s)dB_s - \int_t^T V_s dW_s \\ U_t \geq S_t \\ \int_0^T (U_s - S_s)\, dK_s = 0 \end{cases} \tag{3.3}$$

We also need the following comparison theorem.

Theorem 30 (comparison theorem) *Let* (ξ^1, f^1, g, S^1) *and* (ξ^2, f^2, g, S^2) *be two MF-RBDSDEs. Each one satisfying all the previous assumptions H1), H2), H3) and H4). Assume moreover that :*

i) $\xi^1 \leq \xi^2$ *a.s.*

ii) $f^1(t,y',y,z',z) \leq f^2(t,y',y,z',z)$ $d\mathbb{P} \times dt$ *a.e.* $\forall (y',y,z',z) \in \mathbb{R} \times \mathbb{R}^d$.

iii) $S_t^1 \leq S_t^2,\ 0 \leq t \leq T$ *a.s.*

Let (Y^1, Z^1, K^1) *be a solution of MF-RBDSDE* (ξ^1, f^1, g, S^1) *and* (Y^2, Z^2, K^2) *be a solution of MF-RBDSDE* (ξ^2, f^2, g, S^2). *We suppose also :*

a) One of the two generators is independent of z'.

b) One of the two generators is nondecreasing in y'.

then

$$Y_t^1 \leq Y_t^2, \quad 0 \leq t \leq T \quad a.s.$$

Proof. Suppose that (a) is satisfied by f^1 and (b) by f^2. Applying Itô's formula to $\left|\left(Y_t^1 - Y_t^2\right)^+\right|^2$, and passing to expectation, we have

$$\begin{aligned}
&\mathbb{E}\left|\left(Y_t^1 - Y_t^2\right)^+\right|^2 + \mathbb{E}\int_t^T 1_{\{Y_s^1 > Y_s^2\}}\left|Z_s^1 - Z_s^2\right|^2 ds \\
&= 2\mathbb{E}\int_t^T \left(Y_s^1 - Y_s^2\right)^+ E'\left(f^1\left(s, \left(Y_s^1\right)', Y_s^1, Z_s^1\right) - f^2\left(s, \left(Y_s^2\right)', Y_s^2, \left(Z_s^2\right)', Z_s^2\right)\right) ds \\
&+ 2\mathbb{E}\int_t^T \left(Y_s^1 - Y_s^2\right)^+ \left(dK_s^1 - dK_s^2\right) \\
&\quad + \mathbb{E}\int_t^T \left|E'\left(g\left(s, \left(Y_s^1\right)', Y_s^1, \left(Z_s^1\right)', Z_s^1\right) - g\left(s, \left(Y_s^2\right)', Y_s^2, \left(Z_s^2\right)', Z_s^2\right)\right)\right|^2 1_{\{Y_s^1 > Y_s^2\}} ds.
\end{aligned}$$

Since on the set $\left\{Y_s^1 > Y_s^2\right\}$, we have $Y_t^1 > S_t^2 \geq S_t^1$, then

$$\int_t^T \left(Y_s^1 - Y_s^2\right)^+ \left(dK_s^1 - dK_s^2\right) = -\int_t^T \left(Y_s^1 - Y_s^2\right)^+ dK_s^2 \leq 0$$

Since $f^{\ 1}$ and f^2 are Lipschitz, we have on the set $\{Y_s > Y_s'\}$,

$$\begin{aligned}
&\mathbb{E}\left|\left(Y_t^1 - Y_t^2\right)^+\right|^2 + \mathbb{E}\int_t^T 1_{\{Y_s^1 > Y_s^2\}}\left|Z_s^1 - Z_s^2\right|^2 ds \\
\leq\ &\mathbb{E}\int_t^T \left(6L + \frac{L^2}{1-2\alpha}\right)\left|\left(Y_t^1 - Y_t^2\right)^+\right|^2 + \left|Z_s^1 - Z_s^2\right|^2 ds
\end{aligned}$$

then

$$\mathbb{E}\left|\left(Y_t^1 - Y_t^2\right)^+\right|^2 \leq \mathbb{E}\int_t^T \left(6L + \frac{L^2}{1-2\alpha}\right)\left|\left(Y_t^1 - Y_t^2\right)^+\right|^2 ds$$

The result follows by using Gronwall's lemma. ■

Lemma 31 *i) a.s. for all, t* $Y_t^0 \leq Y_t^n \leq Y_t^{n+1} \leq U_t$.

ii) There exists $Z \in \mathcal{H}^2$ *such that* Z^n *converges to* Z.

Proof. Assertion i) follows from Theorem comparaison. We shall prove ii).

Itô's formula yields

$$\mathbb{E}|Y_0^n|^2 + E\int_0^T |Z_s^n|^2\,ds = \mathbb{E}|\xi|^2 + 2\mathbb{E}\int_0^T Y_s^n \mathbb{E}'\left(f_n(s,(Y_s^n)',Y_s^n,Z_s^n)\right)ds + 2\mathbb{E}\int_0^T S_s dK_s^n$$
$$+\mathbb{E}\int_0^T \mathbb{E}'\left(\left|g(s,(Y_s^n)',Y_s^n,Z_s^n)\right|^2\right)ds$$

From assumption H5) , and the inequality $2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2$ for $\varepsilon > 0$, we get :

$$\mathbb{E}\int_0^T |Z_s^n|^2\,ds \leq \mathbb{E}|\xi|^2 + \frac{LT}{\varepsilon} + \mathbb{E}\int_0^T |g(s,0,0,0)|^2\,ds + (3L\varepsilon + \frac{L}{\varepsilon} + 4L)\mathbb{E}\int_0^T |Y_s^n|^2\,ds$$
$$+\left(\frac{L}{\varepsilon}+\alpha\right)\mathbb{E}\int_0^T \frac{1}{\varepsilon}|Z_s^n|^2\,ds + 2\mathbb{E}\int_0^T S_s dK_s^n$$

On the other hand, we have from (3.2)

$$K_T^n = Y_0^n - \xi - \int_0^T \mathbb{E}' f_n(s,(Y_s^n)',Y_s^n,Z_s^n)ds - \int_0^T \mathbb{E}' g(s,(Y_s^n)',Y_s^n,Z_s^n)dB_s + \int_0^T Z_s^n dW_s \tag{3.4}$$

then

$$\mathbb{E}(K_T^n)^2 \leq C\left(1 + \mathbb{E}\int_0^T \|Z_s^n\|^2 ds\right).$$

We have

$$\begin{aligned} 2\mathbb{E}\int_0^T S_s dK_s^n &\leq \frac{1}{\beta}\mathbb{E}\left(\sup_t |S_t|^2\right) + \beta\mathbb{E}(K_T^n)^2 \\ &\leq \frac{1}{\beta}\mathbb{E}\left(\sup_t |S_t|^2\right) + \beta C\left(1 + E\int_0^T \|Z_s^n\|^2 ds\right), \end{aligned}$$

which yields that

$$\mathbb{E}\int_0^T |Z_s^n|^2\,ds \leq \mathbb{E}|\xi|^2 + \frac{LT}{\varepsilon} + \beta C + \mathbb{E}\int_0^T |g(s,0,0,0)|^2\,ds + (3L\varepsilon + \frac{L}{\varepsilon} + 4L)\mathbb{E}\int_0^T |Y_s^n|^2\,ds$$
$$+\left(\frac{L}{\varepsilon}+\alpha+\beta C\right)\mathbb{E}\int_0^T \frac{1}{\varepsilon}|Z_s^n|^2\,ds + \frac{1}{\beta}\mathbb{E}\left(\sup_t |S_t|^2\right)$$

Choosing ε, β such that $\left(\frac{L}{\varepsilon} + \alpha + \beta C\right) < 1$, we obtain

$$\mathbb{E}\int_0^T \|Z_s^n\|^2 ds \leq C$$

For $n, p \geq K$, Itô's formula gives,

$$\begin{aligned}
&\mathbb{E}(Y_0^n - Y_0^p)^2 + \mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds \\
&= 2\mathbb{E}\int_0^T (Y_s^n - Y_s^p) E'(f_n(s, Y_s^n, (Y_s^n)', Z_s^n) - f_p(s, Y_s^p, (Y_s^p)', Z_s^p))ds \\
&+ 2\mathbb{E}\int_0^T (Y_s^n - Y_s^p) dK_s^n + 2\mathbb{E}\int_0^T (Y_s^p - Y_s^n) dK_s^p \\
&+ \mathbb{E}\int_0^T \left\| E'\left(g(s, Y_s^n, (Y_s^n)', Z_s^n) - g(s, Y_s^p, (Y_s^p)', Z_s^p)\right)\right\|^2 ds.
\end{aligned}$$

But

$$\mathbb{E}\int_0^T (Y_s^n - Y_s^p) dK_s^n = \mathbb{E}\int_0^T (S_s - Y_s^p) dK_s^n \leq 0$$

Similarly, we have $\mathbb{E}\int_0^T (Y_s^p - Y_s^n) dK_s^p \leq 0$.

Therefore,

$$\begin{aligned}
\mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds \leq & 2\mathbb{E}\int_0^T (Y_s^n - Y_s^p)\mathbb{E}'(f_n(s, Y_s^n, (Y_s^n)', Z_s^n) - f_p(s, Y_s^p, (Y_s^p)', Z_s^p))ds \\
&+ \mathbb{E}\int_0^T \left\| \mathbb{E}'\left(g(s, Y_s^n, (Y_s^n)', Z_s^n) - g(s, Y_s^p, (Y_s^p)', Z_s^p)\right)\right\|^2 ds
\end{aligned}$$

By Hôlder's inequality and the fact that g is Lipschitz, we get

$$\begin{aligned}
&\mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds \\
&\leq \left(\mathbb{E}\int_0^T (Y_s^n - Y_s^p)^2 ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T \mathbb{E}'(f_n(s, Y_s^n, (Y_s^n)', Z_s^n) - f_p(s, Y_s^p, (Y_s^p)', Z_s^p))^2 ds\right)^{\frac{1}{2}} \\
&+ L\mathbb{E}\int_0^T \left(|Y_s^n - Y_s^p|^2 + |(Y_s^n)' - (Y_s^p)'|^2\right) ds + \alpha\mathbb{E}\int_0^T |Z_s^n - Z_s^p|^2 ds
\end{aligned}$$

Since $\sup_n \mathbb{E}\int_0^T |f_n(s, Y_s^n, (Y_s^n)', Z_s^n)|^2 \leq C$, we obtain,

$$\mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds \leq C\left(\mathbb{E}\int_0^T (Y_s^n - Y_s^p)^2 ds\right)^{\frac{1}{2}}$$

Hence

$$\mathbb{E}\int_0^T \|Z_s^n - Z_s^p\|^2 ds \longrightarrow 0, \ as \ \ n,p \to \infty$$

Thus $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}^2(\mathbb{R}^d)$. ■

Proof. of theorem 28. Put $Y_t = sup_n Y_t^n$, we have $(Y^n, Z^n) \to (Y, Z)$ in $S^2(\mathbb{R}^d) \times \mathcal{H}^2(\mathbb{R}^d)$. Then, along a subsequence which we still denote (Y^n, Z^n), we get

$$(Y^n, Z^n) \to (Y, Z), \quad dt \otimes d\mathbb{P} \ a.e$$

then, using Lemma 29, we get $f_n(t, Y_t^n, (Y_t^n)', Z_t^n) \to f(t, Y_t, (Y_t)', Z_t) \quad dP \otimes dt \ a.e$

On the other hand, since $Z^n \longrightarrow Z$ in $\mathcal{H}^2(\mathbb{R}^d)$, then there exist $\Lambda \in \mathcal{H}^2(\mathbb{R})$ and a subsequence which we still denote Z^n such that $\forall n, |Z^n| \leq \Lambda$, $Z^n \longrightarrow Z$, $dt \otimes dP \ a.e.$

Moreover from H5), and Lemma 31 we have

$$|f_n(t, Y_t^n, (Y_t^n)', Z_t^n)| \leq K(1 + \sup_n |Y_t^n| + \sup_n |(Y_t^n)'| + \Lambda_t) \in \mathbb{L}^2([0,T], \ dt), \quad P - a.s,$$

It follows from the dominated convergence theorem that,

$$\mathbb{E}\int_0^T \left|\mathbb{E}'\left(f_n(s, Y_s^n, (Y_t^n)', Z_s^n) - f(s, Y_s, (Y_s)', Z_s)\right)\right|^2 ds \longrightarrow 0, \quad n \to \infty. \tag{3.5}$$

We have,

$$\begin{aligned} &\mathbb{E}\int_0^T \|\mathbb{E}'\left(g(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)\right)\|^2 ds \\ &\leq C\mathbb{E}\int_0^T \mathbb{E}'\left(|Y_s^n - Y_s|^2 + |(Y_s^n)' - (Y_s)'|^2\right) ds \\ &+ \alpha\mathbb{E}\int_0^T \|Z_s^n - Z_s\|^2 ds \longrightarrow 0, \quad \text{as } n \to \infty. \end{aligned}$$

It is not difficult to show that (Y, Z) is solution to our MF-RBDSDE. Let

$$\begin{aligned} \overline{Y}_t &= \xi + \int_t^T \mathbb{E}' f\left(s, Y_s, (Y_s)', Z_s\right) ds + K_T - K_t \\ &+ \int_t^T \mathbb{E}' g\left(s, Y_s, (Y_s)', Z_s\right) dB_s - \int_t^T \overline{Z}_s dW_s, \end{aligned}$$

where $\overline{Z} \in \mathcal{H}^2$, $\overline{Y} \in S^2$, $K_T \in \mathbb{L}^2$, $\overline{Y}_t \geq S_t$, (K_t) is continuous and nondecreasing, $K_0 = 0$ and $\int_0^T \left(\overline{Y}_t - S_t\right) dK_t = 0$, and (Y^*, Z^*, K^*) be a solution of (3.1). Then, by Theorem (30), we have for every $n \in \mathbb{N}^*$, $Y^n \leq Y^*$. Therefore, $\overline{Y}$ is a minimal solution of (3.1) ■

Remark 32 *Using the same arguments and the following approximating sequence*

$$h_n(t,x,y,z) = \sup_{(u,v,w)\in\mathbb{Q}^p} \left\{h(u,v,w) - n|x-u| - n|y-v| - n|z-w|\right\},$$

one can prove that the MF-RBDSDE (3.1) has a maximal solution.

Bibliography

[1] Alibert. J., Bahlali. B., Genericity in deterministic and stochastic differential equations. *Séminaire de Probabilités XXXV, Lect. Notes. Math. 1755, 220-240, (2001), Springer Verlag, Berlin-Heidelberg.*

[2] Bahlali. K. Backward stochastic differential equations with locally Lipschitz coefficient. *C.R.A.S*, Paris, serie I Math. 331, 481-486, (2001).

[3] Bahlali. K. Existence, uniqueness and stability for solutions of backward stochastic differential equations with locally Lipschitz. Electron. Comm. Probab., 7, 169-179, (2002).

[4] Bahlali. K., Hassani. M, Mansouri.B, and Mrhardy. N. One barrier reflected backward doubly stochastic differential equations with continuous generator. Comptes Rendus Mathematique, Volume 347, Issue 19, Pages 1201-1206, (2009).

[5] Barles. G, and Lesigne, E. : SDE, BSDE and PDE. Pitman Research Notes in Mathematics Series 364, 47-80 (1997). 364, 47-80 (1997).

[6] Bossy. M, and Talay, D. (1997). A stochastic particle method for the McKean–Vlasov and the Burgers equation. Math. Comp. 66 157–192.

[7] Boufoussi. B, Casteren. J, Mrhardy. N, Generalized backward doubly stochastic differential equations and SPDEs with nonlinear Neumann boundary conditions[J]. Bernoulli, 13: 423-446.(2007).

[8] Buckdahn. R, Li. J, and Peng. S. (2007). Mean-field backward stochastic differential equations and related patial differential equations. Stochastic processes and their applications, 119:3133-3154, (2009).

[9] Buckdahn. R, Djehiche. B, Li. J, Peng. S. Mean-field backward stochastic differential equations. A limit approach. Annals of Probability, 37(4): 1524-1565, (2009).

[10] Briand. Ph, and Carmona. R: BSDEs with polynomial growth generators, J. Appl. Math. stochastic Anal. 13, 207-238 (2000).

[11] Briand. Ph, Delyon. B, Hu. Y, Pardoux. E, Stoica. L: Lp solutions of Backward Stochastic Diffrential Equations, Stochastic Processes and their Applications, 108, 109-129 (2003).

[12] Chan. T. (1994). Dynamics of the McKean–Vlasov equation. Ann. Probab. 22 431–441.

[13] Chaouchkhouane. N, Labed. B, Mansouri. B, Mean-field Reflected Backward Doubly Stochastic DE With Continuous Coefficients. Journal of Numerical Mathematics and Stochastics, 6 (1) : 62-72, 2014

[14] Cvitanic. J, and Karatzas. I: Backward Stochastic Diffrential Equations with reflection and Dynkin Games, Ann. Probab. 24(4), 2024-2056 (1995).

[15] Darling: R. Constructing of Gamma-martingales with prescribed limits, using backward SDE's. Annals of Probability, 1995, vol. 23, p.1234-1261.

[16] Du. H, Li. J, Wei, Q.M: Mean-field backward stochastic differntial equations with continuous coefficients. In proceedings of the 30th Chinese Control Conference, July 22-24, Yantai, China, (2012).

[17] El-Karoui. N, Peng. S, M.C. Quenez: Backward SDE's, finance and optimization. Preprint 1994.

[18] El Karoui. N, Huang. S-J: A general result of existence and uniqueness of Backward Stochastic Differential Equations. in: Backward stochastic differnetial equations (N.El Karoui, L.Mazliakeds, Pitman Research Notes in Mathematics Series vol.364, Longman, 1997, 27-36.

[19] El Karoui. N, Pardoux. E, Quenez. M.C: Reflected backward SDE's and American options. Numerical Method in Finance, Cambridge University Press, pp. 215-231 (1997).

[20] El Karoui. N, Kapoudjian. C, Pardoux. E, Peng. S, Quenez M.C : Reflected Solutions of Backward SDE and Related Obstacle Problems for PDEs. Ann. Probab. Vol. 25, No. 2, 702-737 (1997).

[21] Hamadène. S: Equation différentielles stochastiques rétrogrades : le cas localement lipschitzien, Ann. Inst. Henri Poincaré, 32, 5, 645-659 (1996).

[22] Hamadène. S : Reflected BSDE's with discontinuous Barrier and Applications, Stoch. and Stoc. Reports, 74, 571-596 (2002).

[23] Hamadène. S, Lepeltier. J.-P: "Zero-sum stochastic differential games and backward equations", Systems and Control Letters 24 p.259-263 (1995).

[24] Hamadène. S, Lepeltier. J.-P: Backward equations, stochastic control and zero-sum stochastic differential games, Stochastic and Stochastic Reports 54, 221-231 (1995).

[25] Hamadène. S, Lepeltier. J.-P: Reflected BSDEs and mixed game problem, Stoch. Proc. and their Applications 85 p. 177-188 (2000).

[26] Hamadène. S, Lepeltier. J.-P, Peng. S: BSDE with continuous coefficients and application to Markovian nonzero sum stochastic differential games, in Pitman Research Notes in Mathematics Series 364 (N. El Karoui and L. Mazliak eds.), 1997.

[27] Hamadène. S, Lepeltier. J.-P., Matoussi. A: Double barrier backward SDEs with continuous coefficient, in Pitman Research Notes in Mathematics Series 364 (N. El Karoui and L. Mazliakeds.), (1997).

[28] Karatzas. I, Shreve. S.E: Brownian Motion and Stochastic Calculus, Springer New York (1987).

[29] Kobylanski, M. : Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth, Annals of Probability 28, 558-602 (2000).

[30] Kotelenez, P. (1995). A class of quasilinear stochastic partial differential equations of McKean–Vlasov type with mass conservation. Probab. Theory Related Fields 102 159–188.

[31] Lasry, J.-M. and Lions, P.-L. (2007). Mean field games. Japan J. Math. 2 229–260.

[32] Lepeltier. J.-P., San Martin. J: Backward stochastic differential equations with continuou coefficient, Statist. Probab. lett., 32, 425-430 (1997).

[33] Lepeltier. J.-P., San Martin. J. : Existence of BSDE with Superlinear Quadratic Coefficient, Stochastics and Stochastic Report 63, 227-240 (1998).

[34] Lepeltier. J.-P., Matoussi. A, Xu. M. : Reflected BSDEs under monotonicity and general increasing growth conditions. Advanced Applied Probability, 37, 1-26 (2005).

[35] Li. J. Reflected mean-field backward stochastic differential equations. Approximation and associated nonlinear PDEs. Journal of Mathematical Analysis and Applications. J. Math.Anal.Appl.413: 47–68 (2014).

[36] Lian. D: A class of backward doubly stochastic differential equations with non Lipschitz coefficients. statistics Probability Letters; 79:2223-2229, (2009).

[37] Luo. J, Li. Z. Mean-field reflected backward stochastic differential equations,Statist.Probab.Lett.821961–1968 (2012).

[38] Ma. J., Yong. J. : Forward-Backward Stochastic Diffrential Equations and Their Applications. Lecture Notes in Mathematics 1702. Springer, Berlin (1999).

[39] Méléard. S. (1996). Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models. In Probabilistic Models for Nonlinear Partial Differential Equations (Montecatini Terme, 1995). Lecture Notes in Math. 1627 42–95. Springer, Berlin.

[40] Mao. X. Adapted solutions of Backward Stochastic Differential Equations with Non-Lipschitz Coefficients [J]. Stochastic Process and their Applications, 58: (1995).

[41] N'zi M., Owo.J. Backward doubly stochastic differential equations with discontinuous coefficients. Statistics Probability Letters, 80:903-909 (2009).

[42] Pardoux. E, and Peng. S. : Adapted Solutions of Backward stochastic differential equation, Systems and Control Letters 14, 55-61 (1990).

[43] Pardoux. E., Peng. S. : Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations, in: Stochastic Differential Equations and their Applications (B. Rozovskii and R. Sowers, eds.), Lect. Not. Cont. Inf. Sci., vol.176, Springer, 200-217. (1992).

[44] Pardoux .E., Peng. S: Backward doubly stochastic differential equations and systems of quasilinear SPDEs[J]. ProbabTheory Related Fields, , 98: 209-227 (1994).

[45] Shi.Y., Gu. K., Liu. Comparison theorem of backward doubly stochastic differential equations and applications[J]. Stochastic Analysis and application, 23: 97-110. (2005).

[46] Xu. R. Mean-field backward doubly stochastic differential equations and related partial differential equations and related SPDEs. boundary value problem2012 2012:114.

[47] Xu. R. Mean-field backward doubly stochastic differntial equation and related SPDEs. Boundary Value Problems, 2012:114, a SpringerOpen Journal, (2012).

[48] Zhi. L., Luo. J. Mean-field reflected backward stochastic differential equations. Statistics and Probability Letters, 82, 1961-1968, (2012).