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Dedication

This study is wholeheartedly dedicated to my beloved parents, who have been my sourceof inspiration and gave me strength when I thought of giving up, they continually provide their moral, spiritual, emotional, and financial support. To my brothers, sisters, relatives, mentor, friends, and classmates who shared their words of advice and encouragement to finish this study.

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0.1 General Introduction

As it is known, modern physics rests on two main foundations; the first is that of the general relativistic theory of Albert Einstein which puts us in the theoretical context in order to understand the world in its macro-dimensions: the planets, the stars, the galaxies and the clusters of galaxies or even what is in extra-universe, it explains the force of gravity in the macro-world. It mainly uses Riemannian geometry as a mathematical formalism. The second foundation is that of quantum mechanics, it informs us, in the theoretical framework, about the world in its micro-dimensions: molecules, atoms and even the tiny components of the latter, such as electrons and quarks; and it explains the three main forces in the micro-world. (weak, electromagnetic, and strong forces). It uses the theory of operator algebras acting on a Hilbert space (Von Neumann algebras). The goal of quoting these two foundations each separately is an achievement, but if put together, they become unreliable on all forecasts. We cite as an example the Standard model which has been very successful in mixing the three main interactions. (the strong, electromagnetic and weak interactions), then, there were other models of fusion of the four forces in only one model,such as: the supersymmetric model, the model with extra dimensions, string theory and M-theory.

All these attempts involve or lead to two new concepts: the appearance of a fundamental length (minimum distance) and the emergence of noncommutative geometry. Where the idea is based on the generalization of ordinary canonical commutation relations and Heisenberg's uncertainty principle, which leads to a new non-commutative algebra.

The quantum field theory in curved space via generalizations of the Heisenberg algebra, such as the extended uncertainty principle, is one of the various attempts to include gravity in the quantum world that has attracted a lot of interest. This extended principle allows gravity to be incorporated into quantum mechanics by accounting for the quantum fluctuations of the gravitational field. The existence of a minimum length scale of the Planck order is one of the effects of this unification[1]. By making minor adjustments to the canonical commutation relations and consequently shifting their standard algebra, we can connect this minimum length to a modification of the standard Heisenberg algebra; We refer to

Mignemi's[1] work here, which demonstrated that the Heisenberg relations are altered in the (anti-) de Sitter space by adding corrections proportional to the cosmological constant. Additionally, string theory[2], noncommutative geometry[3], black hole physics[4][5], doubly special relativity (DSR)[6][7], and DSR were used as inspiration for these modifications. The modification of inertia predicted by some alternative theories of gravity at cosmic scales can be naturally derived within the framework of the EUP[8], as recently demonstrated, from the effects of Newton's gravity on quantum systems [9].

The study of relativistic quantum mechanics with the EUP has attracted a lot of research attention in recent years[10][11][12]. Even though the covariant Klein-Fock-Gordon equation in the conventional field theory method of the de Sitter (dS) and anti-de Sitter (AdS) models cannot be used to derive any non-relativistic covariant Schrödinger-like equation, some problems in nonrelativistic quantum mechanics have also been solved [13][14][15].

In addition, there is growing interest in two-dimensional (2D) systems that describe the dynamics of a charged particle contained by a strong and uniform external magnetic field. This fascination is brought on by their numerous uses in various areas of matter physics and chemistry[16][17][18], semiconductor structures [19], chemical physics [20], and molecular vibrational and rotational spectroscopy [21]. As a result, numerous works in both usual and deformed quantum mechanics were devoted to studying this type of problem. Here, we list some examples of 2D systems that are affected by external fields, including the Schrödinger oscillator [22], the KG particle in a pseudo-harmonic oscillator interaction [23], the Dirac equation [24], and the Duffin-Kemmer Petiau (DKP) equation in the cosmic string background [25],the Schrödinger equation with minimal length in noncommutative phase space[26], the KG oscillator with the presence of a minimal length in the noncommutative space[27], the Dirac oscillator in deformed space[28], the relativistic oscillators in a noncommutative space[29] and the bosonic oscillator equation with the Snyder–de Sitter model[30].

During the last years, at the level of a microscopic scale of high energy, many theories have been devoted to the study of the problems of quantum field theory characterized by the nonlocality of the physical processes, to absorb the infinities tainting the standard field theories. Notably, the theory of noncommutative geometry which has been suggested that any unifying scheme of the fundamental interactions of physics should in principle contain effects of the noncommutativity of space describing the nonlocality of quantum phenomena. This is one of the arguments recently proposed by string theory in an attempt to unify physics. In addition, according to mathematician Alain Connes, this non-commutativity of space is considered to be a generalization of the duality between geometric space and algebra to the more general case where algebra is no longer commutative. This leads us to modify two fundamental concepts of mathematics, those of space and symmetry, and to adapt all the mathematical tools, including infinitesimal calculus and cohomology to these new paradigms. In this regard, several works have emerged in the hope of giving this non-commutativity of space a concrete aspect. Their field of application extends from field theory to quantum mechanics. Among these works let us mention the noncommutative Φ^4 theory [31] where the perturbative aspects of the theories

Noncommutative field theory have been widely studied by the calculation of effective actions and two-point Green's functions, the extraction of noncommutative quantum mechanics from noncommutative quantum field theory in the nonrelativistic case [32], and the noncommutative quantum electrodynamics (QED) theory [33] in which Calmet calculated the anomalies of the magnetic spin moments.In addition, this field of application can also extend to the case of relativistic quantum mechanics, for example, the Dirac equation for a particle of spin 1 in interaction with a constant electromagnetic field which has been studied explicitly in noncommutative geometry.[34], where the particle pair production rate has been well determined by the usual Bogolubov method, and the KG and Dirac oscillator problem which has been studied and discussed by Mirza in the noncommutative space[35], where he showed that the problem of the two distorted oscillators has a behavior similar to the Landau problem in a commutative space.

On the other hand, several authors have recently conducted studies of the thermal properties of some quantum systems within the context of usual quantum mechanics and its deformed version. We specifically mention the ordinary Dirac oscillator's thermodynamic features because they describe the quark-gluon plasma model, which has attracted a lot of interest[36][37][38].A linear potential for the Klein-Gordon (KG) and Dirac equations[39], as well as a one-dimensional Schrödinger equation with a harmonic oscillator and an inverse square potential, are also mentioned.[40] A thermal bath in deformed quantum mechanics has also been used to study some aspects of the bosonic oscillator[41][42][43].

The aim of this thesis is to deal mainly with some problems of relativistic quantum mechanics for Bosonic and Fermionic equations, governed by the Klein Gordon,Duffin Kemmer Petiau and Dirac equations respectively In first chapter,we define the studied relativistic equations in this thesis and their properties.In second chapter, we consider the study of the problem of the harmonic oscillator with the interaction of an external magnetic field in the framework of Anti-de Sitter space we are interested in phenomenological models of quantum gravity.

We study analytically in 2D spaces, both KG and DKP (scalar and vector cases) equations in the position space representation for deformed quantum mechanics with EUP, then the same study for Dirac equation and we take a special case for the (2+1)D massless Dirac equation,before doing this,we give a review for the deformed quantum mechanics relations and defining the Nikiforov-Uvarov method to solve the systems, in order to obtain the energy eigenvalues and the corresponding wave functions.To conclude this chapter by analysing the thermodynamic properties of the studied systems.

In the third chapter, we examine the KG and DKP oscillators under the influence of an external magnetic field in the noncommutative space, also,we do the same study for Dirac equation, to sum up this section by studying the thermodynamic properties for both systems.

Fourth chapter deals with both deformations of the two spaces (ie; AdS and NC) on the studied equations KG and Dirac, after that DKP equation and solving the systems as the same process as the second chapter with Nikiforov Uvarov method, at last we evaluate the thermodynamic properties taking into account the two deformations.Concluding this thesis by a general conclusion which involves all the main findings.

Chapter 1

The Relativistic Equations

1.1 Introduction

In the description of relativistic quantum mechanics there are several relativistic wave equations, which are mainly defined according to the spins of the particles whose full spin particles are bosons and those with finite mean spin are fermions. The equations for any spin were considered a subject to study in depth, with the Klein Gordon equation being the appropriate relativistic equation for particles with spin 0 and those with spin 1/2 the mass of the relativistic Dirac equation can be determined, while the Weyl equation spin 1/2 describes massless particles. For the wave equation that describes the Spin 0 and Spin1 particles in a single equation, is the Duffin Kemmer Petiau equation. For relativistic equations that describe the motion of spin3 / 2 particles, there are numerous, such as: the RaritaSchwinger equation, the FiskTait equation, the BhabhaGupta equation ...

The objective of this chapter is to give a detailed information and some notes for the studied relativistic equations such as Klein Gordon,Dirac,and Duffin Kemmer Petiau (DKP) equations in this thesis.

1.2 Klein Gordon equation

The Klein-Gordon equation was established in 1926 by physicists Oskar Klein and Walter Gordon; it is a relativistic wave equation that must converge to the Schrödinger equation in the non-relativistic limit; it is the equation that describes massive particles with zero spin. The Higgs boson (mass> 115 GeV / c^2) is the only fundamental spin-0 particle whose field obeys the Klein-Gordon equation.On the other hand, composite spin-0 particles such as mesons (pions), K mesons (Kaons), and atomic nuclei of specific atoms constituted of an even number of protons and neutrons such as 12C, 16O, 28Si... are described by the Klein-Gordon equation.

We recall that the equation yields the Klein-Gordon equation for a free particle with mass m.

$$E^2 = p^2c^2 + m^2c^4 \tag{1.1}$$

and by considering energy and momentum as differential operators, as is customary

$$p \rightarrow -i\hbar\boldsymbol{\nabla} \quad \text{and } E \rightarrow i\hbar\frac{\partial}{\partial t}, \tag{1.2}$$

we obtain the relativistic wave equation

$$-\hbar^2\frac{\partial^2\Psi}{\partial t^2} = \left(-c^2\hbar^2\boldsymbol{\nabla}^2 + m^2c^4\right)\Psi, \tag{1.3}$$

which, when rearranged, yields

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + \frac{m^2c^2}{\hbar^2}\right)\Psi = 0 \tag{1.4}$$

The Klein-Gordon equation is represented by the equation (1.4), and its complex conjugate is provided by

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + \frac{m^2c^2}{\hbar^2}\right)\Psi^* = 0 \tag{1.5}$$

We obtain by multiplying the eq(1.4)by Ψ^* on the left and eq (1.5) by Ψ on the left and subtracting

$$\frac{1}{c^2}\left(\Psi^*\frac{\partial^2\Psi}{\partial t^2} - \Psi\frac{\partial^2\Psi^*}{\partial t^2}\right) - \Psi^*\boldsymbol{\nabla}^2\Psi + \Psi\boldsymbol{\nabla}^2\Psi^* = 0 \tag{1.6}$$

since

$$\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right) = \Psi^*\frac{\partial^2\Psi}{\partial t^2} - \Psi\frac{\partial^2\Psi^*}{\partial t^2}, \tag{1.7}$$

and

$$\boldsymbol{\nabla}\cdot\left(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right) = \Psi^*\boldsymbol{\nabla}^2\Psi - \Psi\boldsymbol{\nabla}^2\Psi^*, \tag{1.8}$$

Eq(1.6)can be represented as a continuity equation.

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0, \tag{1.9}$$

Using the same expression for the current density **J**

$$\mathbf{J} = \frac{\hbar}{2mi}\left\{\Psi^* \boldsymbol{\nabla}\Psi - \left(\boldsymbol{\nabla}\Psi^*\right)\Psi\right\} \tag{1.10}$$

as given for the non-relativistic Schrodinger equation.When we choose as the probability density ρ, the continuity equation will be satisfied.

$$\rho = \frac{i\hbar}{2mc^2}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right). \tag{1.11}$$

The probability density ρ provided by the eq(1.11) involves both Ψ and $\frac{\partial\Psi}{\partial t}$ which can be arbitrarily determined and so accepts both positive and negative values.Because the probability density must be a positive definite quantity, the Klein-Gordon equation was rejected as a wave equation for many years until Pauli and Weisskopf reinterpreted it as a field equation in the same way that Maxwell's equation for electromagnetic field was. By inserting the rest mass $m = 0$ into eq(1.4), we obtain the electromagnetic field field equation.

The Klein-Gordon equation is a second-order differential equation in t, and it has produced physically undesirable negative values for the probability density ρ as well. It can be seen that the Schrodinger equation is a first-order differential equation in t, and hence the probability density has a positive definite value. Using this hint, Dirac attempted to linearize the relativistic relation $E^2 = p^2c^2 + m^2c^4$, which is quadratic in both E and p, and thus discovered the Dirac equation.

Thus, efforts to overcome early challenges in the formulation of Relativistic Quantum Mechanics worked out great. Dirac was successful in linearizing the relativistic relation (1.1), which is quadratic in both energy and momentum, and obtaining the Dirac equation for the electron, which has intrinsic spin and magnetic moment features. The interpretation of the Klein-Gordon equation as a field equation sowed the seeds of the Quantum Field theory.[44]

1.3 Dirac equation

We follow the historical path taken by Dirac in 1928, looking for a relativistic covariant equation of the form $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ with a positive definite probability density. Since such an equation is linear in the derivative of time, it is natural to try to form a Hamiltonian in the derivative of space. This equation can take a form

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\left(\alpha_1\frac{\partial\psi}{\partial x^1} + \alpha_2\frac{\partial\psi}{\partial x^2} + \alpha_3\frac{\partial\psi}{\partial x^3}\right) + \beta mc^2\psi \equiv H\psi \tag{1.12}$$

Coefficients are still unknown α_i cannot be a simple number, otherwise eq(1.12) would have no invariant form for simple spatial rotation. we suspect that α_i is a matrix and represent it using the sign operator $\wedge$. Then ψ cannot be a simple scalar quantity.in fact, the probability density $\rho = \psi^*\psi$ must be the temporal component of a conservative vector at four if its integral over all space, at fixed t, is an invariant.

To free the eq(1.12) of these constraints, Dirac proposed to treat it as a matrix equation. ψ wave function, similar to the spin wave functions of uncorrelated quantum mechanics, is written as a column matrix with N components

$$\psi = \begin{pmatrix} \psi_1(x,t) \\ . \\ . \\ \psi_N(x,t) \end{pmatrix} \tag{1.13}$$

and the constant coefficients α_i,β are$N \times N$ matrices. Therefore,eq (1.12) is replaced by Npaired first order equations.

$$\begin{aligned} i\hbar\frac{\partial\psi_\sigma}{\partial t} &= \frac{\hbar c}{i}\sum_{\tau=1}^{N}\left(\alpha_1\frac{\partial}{\partial x^1} + \alpha_2\frac{\partial}{\partial x^2} + \alpha_3\frac{\partial}{\partial x^3}\right)_{\sigma\tau}\psi_\tau + \sum_{\tau=1}^{N}\beta_{\sigma\tau}mc^2\psi_\tau \\ &\equiv \sum_{\tau=1}^{N} H_{\sigma\tau}\psi_\tau \end{aligned} \tag{1.14}$$

Then we apply the matrix notation and remove the summation indices, in which case eq(1.14) comes out as eq(1.12), now interpreted as a matrix equation.

If this equation is to serve as a satisfactory starting point, it must first give the correct energy-momentum relationship

$$E^2 = p^2c^2 + m^2c^4 \tag{1.15}$$

For a free particle, second, it must allow for a continuity equation and explain the probability for the ψ wavefunction, and third, it must be the Lorentz covariance. We now discuss the first two of these requirements.

For the correct energy-momentum relationship to emerge from the eq(1.12), each ψ_σ component of ψ must satisfy Klein Gordon's quadratic equation, or

$$-\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \psi_\sigma \tag{1.16}$$

Iterating eq (1.12) ,we find

$$\begin{aligned} -\hbar^2 \frac{\partial^2 \psi}{\partial t^2} &= -\hbar^2 c^2 \sum_{i,j=1}^{3} \frac{\alpha_j \alpha_i + \alpha_i \alpha_j}{2} \frac{\partial^2 \psi}{\partial x^i \partial x^j} \\ &+ \frac{\hbar m c^3}{i} \sum_{i=1}^{3} \left(\alpha_i \beta + \beta \alpha_i\right) \frac{\partial \psi}{\partial x^i} + \beta^2 m^2 c^4 \psi \end{aligned} \tag{1.17}$$

We can revive the eq (1.16) if the four matrices α_i , β obey algebra

$$\begin{aligned} \alpha_i \alpha_j + \alpha_j \alpha_i &= 2\delta_{ij} \\ \alpha_i \beta + \beta \alpha_i &= 0 \\ \alpha_i^2 = \beta^2 &= 1 \end{aligned} \tag{1.18}$$

What other properties do we require from these four matrices α_i, β and can we construct them explicitly? α_i and β must be Hermitian matrices so that Hamilton H in eq (1.14) is a postulately chosen Hermitian operator. Since eq (1.18), $\alpha_i^2 = \beta^2 = 1$, the eigenvalues of α_i and β are ± 1. In addition, it results from their anticommutation properties that the trace, i.e. sum of diagonal elements, of each α_i and β is zero. For example

$$\alpha_i = -\beta \alpha_i \beta \tag{1.19}$$

and according to the cyclical nature of the trace

$$Tr\ AB = Tr\ BA \tag{1.20}$$

one has

$$Tr\ \alpha_i = +Tr\ \beta^2 \alpha_i = +Tr\ \beta \alpha_i \beta = -Tr\ \alpha_i = 0 \tag{1.21}$$

Since the trace is just the sum of eigenvalues, the number of positive and negative eigenvalues ± 1 must be equal, so α_i and β must be matrices of even dimensions. The smallest even dimension $N = 2$ is excluded, because it can only accommodate three mutually anticommutating Pauli matrices σ_i plus one identity matrix. The smallest dimension that can be made for α_i and β is $N = 4$, which is the situation we will study. in a particular unambiguous representation, the matrices are

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.22}$$

where the σ_i are the familiar 2×2Pauli matrices and the unit entries in β stand for 2×2 unit matrices,we have in detail

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.23}$$

In order to construct the law of conservation of current differential, we first introduce the Hermitian conjugate wave function $\psi^\dagger = (\psi_1^*, .., \psi_4^*)$ and left-multiply eq(1.21)by $\psi^\dagger$

$$i\hbar\psi^\dagger \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \sum_{k=1}^{3} \psi^\dagger \alpha_k \frac{\partial}{\partial x^k} \psi + mc^2 \psi^\dagger \beta \psi \tag{1.24}$$

Next, we form the Hermitian conjugate of eq(1.12) and multiply it to the right by ψ

$$-i\hbar \frac{\partial \psi^\dagger}{\partial t} \psi = -\frac{\hbar c}{i} \sum_{k=1}^{3} \frac{\partial \psi^\dagger}{\partial x^k} \alpha_k \psi + mc^2 \psi^\dagger \beta \psi \tag{1.25}$$

where $\alpha_i^\dagger = \alpha_i, \beta^\dagger = \beta$.Substracting eq(1.25)from eq(1.24), we obtain

$$i\hbar \frac{\partial}{\partial t} \psi^\dagger \psi = \sum_{k=1}^{3} \frac{\hbar c}{i} \frac{\partial}{\partial x^k} \left(\psi^\dagger \alpha_k \psi \right) \tag{1.26}$$

or

$$\frac{\partial}{\partial t} \rho + \operatorname{div} j = 0 \tag{1.27}$$

where we perform the determination of the probability density

$$\rho = \psi^{\dagger}\psi = \sum_{\sigma=1}^{4} \psi_{\sigma}^{*}\psi_{\sigma} \tag{1.28}$$

And the probability flow with three components

$$j^{k} = c\psi^{\dagger}\alpha^{k}\psi \tag{1.29}$$

by integrating eq(1.27)over all space and using Green's theorem, we find

$$\frac{\partial}{\partial t}\int d^{3}x\psi^{\dagger}\psi = 0 \tag{1.30}$$

This supports the possibility of interpreting $\rho = \psi^{\dagger}\psi$as a definite probability density.

Notation eq(1.27) predicts that the current probability J forms a vector if eq(1.29) is invariant in three-dimensional rotation. We really need to show more than that. The density and the current in eq(1.27) must form four vectors under the Lorentz transformation to ensure the covariance of the continuity equation and the probabilistic interpretation. Moreover, it is necessary to show that the Dirac equation eq (1.12) is a Lorentz covariance before it can be considered as satisfied[44][45].

1.4 DKP equation

The Duffin-Kemmer-Petiau equation describes the dynamics of scalar and vector particles with spins of 0 and 1, respectively. It is covariant, first-order relative to time, similar to Dirac's. It follows a more complex algebra, with three irreducible representations, a one-dimensional trivial representation, five dimensions related to spin 0, and 10 dimensions related to spin 1.

Furthermore, this DKP equation contains all the information of the 0-spin scalar particle system, but it is not completely equivalent to the Klein-Gordon (KG) equation, except for the absence of interaction. Otherwise, its quadratic form contains an additional non-physical term, which may be due to the mixing of spin sectors 0 and 1, which is different from Dirac's.Perhaps this equation has remained unimportant years because of this major defect.

Now, equality is established by showing this contradiction becomes clear and can be revealed by the correct interpretation of the DKP theory. Recently, this theory has generated a particular interest, which has become the main mission of the quantum dynamics of the other party and has undergone tremendous development. In this regard, many problems such as many problems have been resolved or a central field with spin has been resolved. Curve space time; External field of gravity; Potential step. ..Etc.

In this section we propose to give the essential factors and representations related to scalar bosons and vectors of DKP which will be useful to us later, from the DKP equation.

$$[i\beta^{\mu}D_{\mu}-m]\,\psi(r)=0 \tag{1.31}$$

with $D_{\mu}=\partial_{\mu}+ieA_{\mu}$and β^{μ} are singular matrices verifying the following commutation relationships

$$\beta^{\mu}\beta^{\nu}\beta^{\lambda}+\beta^{\lambda}\beta^{\nu}\beta^{\mu}=g^{\mu\nu}\beta^{\lambda}+g^{\nu\lambda}\beta^{\mu} \tag{1.32}$$

Where the auxiliary of the eq(1.31) is given by

$$[i\left(\partial_{\mu}-ieA_{\mu}\right)]\,\overline{\psi}(x,t)\beta^{\mu}+m\overline{\psi}(x,t)=0 \tag{1.33}$$

here $\overline{\psi}=\psi^{+}\left(2\beta_{0}^{2}-1\right).$

From eq(1.31) and eq(1.33), it is easy to obtain the following continuity equation:

$$\partial_{\mu}J^{\mu}=0 \tag{1.34}$$

where $J^{\mu}=\overline{\psi}\beta^{\mu}\psi$.

1)- For spin 0(scalar case), the associated representation is 5 dimensions, whose matrices are given explicitly by

$$\beta^{0}=\begin{pmatrix}\boldsymbol{\theta} & \widetilde{\mathbf{0}}\\ \widetilde{\mathbf{0}}_{\mathbf{T}} & \mathbf{0}\end{pmatrix},\ \text{and}\ \beta\equiv\beta^{i}=\begin{pmatrix}\widehat{\mathbf{0}} & \rho^{i}\\ -\rho_{T}^{i} & \mathbf{0}\end{pmatrix},i=1,2,3 \tag{1.35}$$

with, $\mathbf{0}$, $\widetilde{\mathbf{0}}$ are respectively null matrices of dimensions 2×2, 2×3;and

$$\boldsymbol{\theta}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\rho^{1}=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\rho^{2}=\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } \rho^{3}=\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.36}$$

2)- In the second case of a spin 1 particle (vector case), the associated representation is 10 dimensions and the matrices β^{μ}, are given by

$$\beta^{0}=\begin{pmatrix} 0 & \overline{0} & \overline{0} & \overline{0} \\ \overline{0}^{T} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \overline{0}^{T} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \overline{0}^{T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \text{and } \beta\equiv\beta^{i}=\begin{pmatrix} 0 & \overline{0} & e_{i} & \overline{0} \\ \overline{0}^{T} & \mathbf{0} & \mathbf{0} & -is_{i} \\ -e_{i}^{T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \overline{0}^{T} & -is_{i} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{1.37}$$

$, i=1,2,3$

where the s_i matrices are the usual matrices (3×3) of spin 1, which are defined as follows:

$$s_{1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, s_{2}=\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, s_{3}=\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.38}$$

and $\mathbf{0}$ and $\mathbf{1}$, are respectively the null matrix and the matrix unit of dimensions (3×3) and the matrices $\overline{0}$ and e_i are defined as follows [46]:

$$\overline{0}=(000)\,, e_{1}=(100)\,, e_{2}=(010)\,, e_{3}=(001) \tag{1.39}$$

Chapter 2

Relativistic oscillators in Anti de-Sitter space

2.1 Introduction

Anti-de Sitter space-time is one of the simplest and most symmetric solutions to Einstein's field equations, including the cosmological constant. After 1998, this space-time drew the attention of high energy physicists due to the proposed anti-de Sitter space/conformal field theory (Ads/CFT). AdS is an exact solution of Einstein's general relativity field equations, characterized by a constant negative space-time curvature and a vanishing stress-energy tensor (vacuum solution). It is a near relative of de Sitter space, a vacuum solution with a continuous positive space-time curvature. In general, the cosmological constant in AdS is negative, except in the lowest dimensional case, where it is zero.This is in contrast to the actual universe, in which the stress-energy tensor does not vanish and seems to be small but positive, and hence AdS is insufficient to describe our physical universe. On the other hand,AdS can still yield insights into the nature of the universe.

The objective of this chapter is to study analytically the two dimensional deformed relativistic oscillators equations subjected to the effect of a uniform magnetic field. And we consider the presence of a minimal uncertainty in momentum caused by the anti-de Sitter model so that we present a review of the deformed quantum mechanics relations and we show the Nikiforov Uvarov (NU) method to solve the systems. Finally, we evaluate the thermodynamic properties of the systems.

2.2 Review of the deformed quantum mechanics relation

The deformed Heisenberg algebra leading to the EUP in AdS model is defined in 3D spaces by the following commutation relations [47][48]:

$$[X_i, X_j] = 0\ ,\ \ [P_i, P_j] = -i\hbar\lambda\epsilon_{ijk}L_k\ ,\ \ [X_i, P_j] = i\hbar\left(\delta_{ij} - \lambda X_i X_j\right) \tag{2.1}$$

where λ is a small positive deformation parameter. In the sense of quantum gravity, this λ parameter is calculated as the fundamental constant associated with the scale factor of the expanding universe and it is proportional to the cosmological constant $\Gamma = -3\lambda = -3a^{-2}$ where a is the AdS radius [49].

L_k are the usual angular momentum components and they are expressed as follows:

$$L_k = \epsilon_{ijk} X_i P_i \tag{2.2}$$

These components follow the usual momentum algebra:

$$[L_i, P_j] = i\hbar\varepsilon_{ijk}P_k\ ,\ [L_i, X_j] = i\hbar\varepsilon_{ijk}X_k\ , [L_i, L_j] = i\hbar\varepsilon_{ijk}L_k \tag{2.3}$$

The AdS deformed algebra (2.1) gives rise to modified Heisenberg uncertainty relations:

$$\Delta X_i \Delta P_i\ \geq \frac{\hbar}{2}\left(1 + \lambda\left(\Delta X_i\right)^2\right) \tag{2.4}$$

where we have choosen the states for which $\langle X_i \rangle = 0$.

It also generates a minimum uncertainty in momentum. For simplicity, if we assume isotropic uncertainties $X_i = X$, we get:

$$(\Delta P_i)_{min} = \hbar\sqrt{\lambda} \tag{2.5}$$

So the noncommutative operators X_i and P_i satisfy the AdS algebra (2.1) with the rescaled uncertainty relations in position space (2.4). In what follows, we represent these operators as functions of the usual x_i and p_i operators fulfilling the ordinary canonical commutation relations in position space; this is done with the following transformations:

$$X_i = \frac{x_i}{\sqrt{1 - \lambda r^2}} \tag{2.6}$$

$$P_i = -i\hbar\sqrt{1 - \lambda r^2}\partial x_i \tag{2.7}$$

Here the variable r vary in the domain $\left]-1/\sqrt{\lambda}, 1/\sqrt{\lambda}\right[$.

2.3 Nikiforov-Uvarov method

Primarily, the Nikiforov-Uvarov (NU) approach was built on the hypergeometric differential equation. The formulas used in the NU method reduce the second order differential equations to the hypergeometric kind with a suitable coordinate transformation (note that $s \equiv s(x)$ and the primes denote the derivatives):

$$\psi^{''}(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\psi^{'}(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{2.8}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are at most second degree polynomials while the degree of the polynomial $\widetilde{\tau}\ (s)$ is strictly less than 2 [50][51]. If we use the following factorization:

$$\psi(s) = \phi(s)y(s) \tag{2.9}$$

eq.(2.8) becomes [51]:

$$\sigma(s)y^{''}(s) + \tau(s)y^{'}(s) + \Lambda y(s) = 0 \tag{2.10}$$

where:

$$\pi(s) = \sigma(s)\frac{d}{ds}(\ln\phi(s)) \text{ and } \tau(s) = \widetilde{\tau}(s) + 2\pi(s) \tag{2.11}$$

and Λ is defined by:

$$\Lambda_n + n\tau^{'} + \frac{n(n+1)}{2}\sigma^{''} = 0 \text{ and } n = 0, 1, 2, ... \tag{2.12}$$

The energy eigenvalues of the system are determined from the equation above. To find them, we have to evaluate $\pi(s)$ and Λ first by identifying:

$$k = \Lambda - \pi^{'}(s) \tag{2.13}$$

We get the solution of the quadratic equation for $\pi(s)$ which is a polynomial of s:

$$\pi(s) = \left(\frac{\sigma^{'} - \widetilde{\tau}}{2}\right) \pm \sqrt{\left(\frac{\sigma^{'} - \widetilde{\tau}}{2}\right)^2 - \widetilde{\sigma} + \sigma k} \tag{2.14}$$

It must be noted that in the calculation of $\pi(s)$, the determination of k is the critical point and it is achieved by stating that the expression under the square root in (2.14) must be a polynomial square; this gives us a quadratic general equation for k. We use (2.11) and the Rodrigues relation to evaluate the polynomial solutions $y_n(s)$:

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} \left[\sigma^n(s)\rho(s)\right] \tag{2.15}$$

where C_n are constants used to normalize the solutions. The weight function $\rho(s)$ meets the following relation:

$$\frac{d}{ds} \left[\sigma(s)\rho(s)\right] = \tau(s)\rho(s) \tag{2.16}$$

This last equation refers to the classical orthogonal polynomials and we write the orthogonality relations for the polynomial solutions as follows:

$$\int_a^b y_n(s) y_m(s) \rho(s) ds = 0 \text{ if } m \neq n \tag{2.17}$$

2.4 Klein Gordon oscillator in a magnetic field

Now, we introduce the equation in the presence of harmonic oscillator with a constant magnetic field in a 2D space [44]:

$$c^2 \left(\mathbf{p} - \frac{e}{c}\mathbf{A} + im\omega\mathbf{r}\right) \cdot \left(\mathbf{p} - \frac{e}{c}\mathbf{A} - im\omega\mathbf{r}\right) \Psi\left(\mathbf{r}\right) = \left(E^2 - m^2c^4\right) \Psi\left(\mathbf{r}\right) \tag{2.18}$$

We choose the z-axis as the magnetic field direction and use the Coulomb gauge:

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \frac{B}{2}\left(-y, x, 0\right) \tag{2.19}$$

Where B represents the intensity of the magnetic field.

We include the AdS algebra definition (eqs.2.6 and (2.7)) to rewrite this equation in the deformed momentum space:

$$c^2(\mathbf{p}^+ \cdot \mathbf{p}^-)\Psi\left(\mathbf{r}\right) = (E^2 - m^2c^4)\Psi\left(\mathbf{r}\right) \tag{2.20}$$

with the following definitions:

$$\mathbf{p}^{\pm} = \mathbf{p}^{'} \pm im\omega \frac{\mathbf{r}}{\sqrt{1-\lambda r^2}}, \mathbf{p}^{'} = \sqrt{1-\lambda r^2}\mathbf{p} - \frac{e}{c}\mathbf{B} \times \frac{\mathbf{r}}{\sqrt{1-\lambda r^2}} \tag{2.21}$$

We get the following equation after a straightforward computation:

$$\left[(1-\lambda r^2)p^2+\eta\frac{r^2}{1-\lambda r^2}+i\hbar\lambda(\mathbf{r}\cdot\mathbf{p})-\frac{eB}{c}L_z-\varepsilon\right]\Psi\left(\mathbf{r}\right)=0 \tag{2.22}$$

with the parameters:

$$\eta=m^2\omega^2+\frac{e^2B^2}{4c^2}-\lambda\hbar m\omega \text{ and } \varepsilon=\frac{(E^2-m^2c^4)}{c^2}+2m\omega\hbar \tag{2.23}$$

To get the exact solution of eq.(2.22), we use the polar coordinates in position space (r,φ) and write the solutions in a seperate form containing the azimuthal quantum number l:

$$\Psi\left(r,\varphi\right)=\exp(il\varphi)R(r), l=0,1,2,... \tag{2.24}$$

So, eq.(2.22) transforms to the following expression:

$$\left[\left(\sqrt{1-\lambda r^2}\frac{d}{dr}\right)^2+\frac{1-\lambda r^2}{r}\frac{d}{dr}-\frac{l^2\left(1-\lambda r^2\right)}{r^2}-\frac{\eta r^2}{\hbar^2\left(1-\lambda r^2\right)}+\frac{\varepsilon}{\hbar^2}+\frac{eBl}{c\hbar}\right]R(r)=0 \tag{2.25}$$

Now to solve eq.(2.25), we use the following transformation:

$$R(\rho)=\rho^{\mu}g\left(\rho\right) \text{ and } \rho=\sqrt{1-\lambda r^2} \tag{2.26}$$

with

$$\frac{d}{d\rho}R(\rho) = \rho^{\mu}\left[\frac{d}{d\rho}+\frac{\mu}{\rho}\right]g\left(\rho\right) \tag{2.27}$$

$$\frac{d^2}{d\rho^2}R(\rho) = \rho^{\mu}\left[\frac{d^2}{d\rho^2}+\frac{2\mu}{\rho}\frac{d}{d\rho}+\frac{\mu\left(\mu-1\right)}{\rho^2}\right]g\left(\rho\right) \tag{2.28}$$

By replacing the eqs.(2.27 and 2.28)into eq.(2.25) ,it gives us:

$$\left[\left(1-\rho^2\right)\frac{d^2}{d\rho^2}+2\left(\frac{\mu}{\rho}-\left(\mu+1\right)\rho\right)\frac{d}{d\rho}-\frac{l^2\rho^2}{\left(1-\rho^2\right)}+\epsilon\right]g\left(\rho\right)=0 \tag{2.29}$$

with

$$\epsilon=\frac{1}{\lambda}\left(\frac{\varepsilon}{\hbar^2}+\frac{eBl}{c\hbar}\right)-2\mu \tag{2.30}$$

where we have chosen the free parameter μ so that it satisfies the relation:

$$\mu\left(\mu-1\right)-\frac{\eta}{\hbar^2\lambda^2}=0 \tag{2.31}$$

The solutions of this equation are given by:

$$\mu = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{\eta}{\hbar^2 \lambda^2}} \tag{2.32}$$

From eq.(2.29), we see that $g(\rho)$ should be nonsingular at $\rho = \pm 1$ and the same is true for $R(\rho)$ from eq.(2.26); thus the accepted value of μ is:

$$\mu = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\eta}{\hbar^2 \lambda^2}} \tag{2.33}$$

We note that eq.(2.29) possesses three singular points $\rho = 0, 1, -1$ and to reduce it to a class of known differetial equation with a polynomial solution, we use another change of variable $s = 2\rho^2 - 1$,with

$$\begin{aligned} ds = 4\rho d\rho \Rightarrow \frac{ds}{d\rho} = 4\rho \\ \frac{d}{d\rho}\left(\frac{d}{d\rho}\right) = \frac{d}{d\rho}\left(4\rho\frac{d}{ds}\right) = 16\rho^2\frac{d^2}{ds^2} + 4\frac{d}{ds} \end{aligned} \tag{2.34}$$

replacing eq(2.34) into eq (2.29),we get;

$$\left[\frac{d^2}{ds^2} + \frac{\left(\mu - \frac{1}{2}\right) - \left(\mu + \frac{3}{2}\right)s}{1 - s^2}\frac{d}{ds} - \frac{\left(l^2 + \epsilon\right)s^2 + 2l^2 s - \left(\epsilon - l^2\right)}{4\left(1 - s^2\right)^2}\right] g(s) = 0 \tag{2.35}$$

Comparing eq.(2.35) with eq.(2.8) allows us to use the NU method with polynomials given by:

$$\begin{aligned} \sigma(s) &= 1 - s^2, \widetilde{\sigma}(s) = \frac{1}{4}\left[-\left(l^2 + \epsilon\right)s^2 - 2l^2 s + \left(\epsilon - l^2\right)\right] \\ \&\ \widetilde{\tau}(s) &= \left(\mu - \frac{1}{2}\right) - \left(\mu + \frac{3}{2}\right)s \end{aligned} \tag{2.36}$$

We replace them in eq.(2.14) to get:

$$\begin{aligned} \pi(s) = \frac{\left(\mu - \frac{1}{2}\right)(s-1)}{2} \pm \Bigg[\frac{1}{2}\left(\left(\mu - \frac{1}{2}\right)^2 + l^2 - (4k - \epsilon)\right)s^2 - \\ \left(\left(\mu - \frac{1}{2}\right)^2 - l^2\right)s + \frac{1}{2}\left(\left(\mu - \frac{1}{2}\right)^2 + l^2 + (4k - \epsilon)\right)\Bigg]^{\frac{1}{2}} \end{aligned} \tag{2.37}$$

The parameter k is determined as mentioned in the precedent section and we get two values:

$$k_1 = \frac{\epsilon}{4} + \frac{l}{2}\left(\mu - \frac{1}{2}\right) \text{ and } k_2 = \frac{\epsilon}{4} - \frac{l}{2}\left(\mu - \frac{1}{2}\right) \tag{2.38}$$

For $\pi(s)$, we obtain the following solutions:

$$\pi(s) = \begin{cases} \pi_{1,3} = \frac{1}{2}\left[(2\mu \mp l + 1)\, s - (2\mu \pm l - 1)\right] \\ \pi_{2,4} = \pm\frac{l}{2}\,(s+1) \end{cases} \tag{2.39}$$

where π_1 and π_2 are related to k_1 while π_3 and π_4 are linked to k_2.

In our case, the relevant solution is the proper value π_4, so that we have:

$$\tau(s) = -\left(\mu + \frac{3}{2} + l\right) s + \left(\mu - \frac{1}{2} - l\right) \tag{2.40}$$

From eqs.(2.12) and (2.13), we obtain:

$$\Lambda_n = k_2 - \frac{l}{2} = n\left(n + \mu + l + \frac{1}{2}\right), n = 0, 1, 2, ... \tag{2.41}$$

Hence, we found the expressions of the energy eigenvalues as:

$$E_{n,l} = \pm mc^2 \left[1 - \frac{2\omega\hbar}{mc^2} + \frac{2\hbar}{mc^2}\left\{(2n+l+1)\sqrt{\left(\omega - \frac{\lambda\hbar}{2m}\right)^2 + \widetilde{\omega}^2} + \frac{\lambda\hbar}{2m}\left(4n\left(n+l+1\right)+2l+1\right) - \widetilde{\omega}l\right\}\right]^{\frac{1}{2}} \tag{2.42}$$

where we have used the definition $\widetilde{\omega} = eB/2mc = \omega_c/2c$ with ω_c the cyclotron frequency.

We notice that the energy spectrum of our system contains two corrections associated with the deformation; the first one comes with the oscillator term and an additional one which increases with the deformation parameter λ. Here it should be noted that, according to the n^2 dependence of the energy levels, which corresponds to a confinement at the high energy area, our result is equivalent to the energy of a spinless relativistic quantum particle in a square well potential; in our case, the boundaries of the well are placed at $\pm\pi/2\sqrt{\lambda}$.

We can test the shape of the energy spectrum as follows: It corresponds to same spectrum of the deformed 2D KG oscillator under a uniform magnetic field with Snyder-de Sitter algebra if we ignore the effect of Snyder algebra ($\alpha_2 = 0$ and $\alpha_1 = \lambda$)[30] and, if we study the limit $\lambda \to 0$, we obtain the exact result of the 2D KG oscillator under a magnetic field without deformation. We also note that the result is strictly consistent with the usual KG oscillator when both the deformed parameter and the magnetic field are absent (i.e. $\lambda = B = 0$) [52].

To deduce the complete expression of the wave functions$R(r)$, we use the relations of $\pi_4(s)$ as follows. We first get

$$\pi(s) = \pi_4(s) = \sigma(s)\frac{d}{ds}(\ln\phi(s)) \Longrightarrow \phi(s) = \exp\left(\int \frac{\pi(s)}{\sigma(s)}ds\right) \tag{2.43}$$

After the calculation of the integral we obtain the function $\phi(s)$ as

$$\phi(s) = (1-s)^{l/2} \tag{2.44}$$

We use (2.16) to find the weight function $\rho(s)$

$$\frac{d}{ds}[\sigma(s)\rho(s)] = \tau(s)\rho(s) \Longrightarrow \int \frac{d\rho(s)}{\rho(s)} = \int\left(\frac{\tau(s)}{\sigma(s)} - \frac{d\sigma(s)}{\sigma(s)\,ds}\right)ds \tag{2.45}$$

When we compute the integral we get

$$\ln\rho(s) = \int\left(\frac{\tau(s)}{\sigma(s)} - \frac{d\sigma(s)}{\sigma(s)\,ds}\right)ds \Longrightarrow \rho(s) = \exp\left[\int\left(\frac{\tau(s)}{\sigma(s)} - \frac{d\sigma(s)}{\sigma(s)\,ds}\right)ds\right] \tag{2.46}$$

We substitute by the expression of $\tau(s)$equation (2.40) and $\sigma(s)$ equation (2.36) ,after that we calculate the integral,we find

$$\rho(s) = (1+s)^{\left(\mu-\frac{1}{2}\right)}(1-s)^{l} \tag{2.47}$$

The $y_n(s)$part is given by Rodrigues relation as follows;

$$y_n(s) = \frac{C_n}{\rho(s)}\frac{d^n}{ds^n}\left[\left(1-s^2\right)^n\rho(s)\right] \tag{2.48}$$

By substituting the expression of $\rho(s)$ from equation (2.47)

$$y_n(s) = \frac{C_n}{\rho(s)}\frac{d^n}{ds^n}\left[\left(1-s^2\right)^n(1+s)^{\left(\mu-\frac{1}{2}\right)}(1-s)^{l}\right] \tag{2.49}$$

We see that eq.(2.48) stands for the Jacobi polynomials:

$$y_n(s) \equiv P_n^{(l,\mu-1/2)}(s) \tag{2.50}$$

Hence, $f(s)$ can be written in the following form:

$$f(s) = C_n(1-s)^{\frac{l}{2}}P_n^{(l,\mu-1/2)}(s) \tag{2.51}$$

In terms of the variables r and φ, we can now write the general form of the wave function Ψ:

$$\Psi\left(r,\varphi\right) = C_n 2^{\frac{l}{2}} e^{il\varphi}\left(1-\lambda r^2\right)^{\frac{\mu}{2}}\left(\lambda r^2\right)^{\frac{l}{2}} P_n^{(l,\mu-1/2)}\left(1-2\lambda r^2\right) \tag{2.52}$$

where μ is given in eq.(2.33) and the constant C_n is obtained using the normalization condition in the space of the radial wave function [48]:

$$\int_0^{1/\sqrt{\lambda}} \frac{2^{l+1}\pi r dr}{\left(1-\lambda r^2\right)^{1/2}} R^*(r)R(r) = 1 \tag{2.53}$$

and the orthogonality relation of the Jacobi polynomials [53]:

$$\int_{-1}^{1} dy(1-y)^a(1+y)^b\left[P_n^{(a,b)}(y)\right]^2 = \frac{2^{a+b+1}\Gamma(a+n+1)\Gamma(b+n+1)}{n!(a+b+1+2n)\Gamma\left(a+b+n+1\right)} \tag{2.54}$$

So we get the normalization constant value:

$$C_n = \sqrt{\frac{\lambda}{2^l\pi}\frac{n!\left(2n+\mu+l+\frac{1}{2}\right)\Gamma\left(n+\mu+l+\frac{1}{2}\right)}{\Gamma\left(n+\mu+\frac{1}{2}\right)\Gamma\left(n+l+1\right)}} \tag{2.55}$$

Let us now check these solutions by studying the limits $\lambda \to 0$ of the expression (2.52). We use the following relations [53]:

$$\lim_{\mu\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\mu}\right) = L_n^{\alpha}\left(x\right) \text{ and } \lim_{\nu\to\infty}\frac{\Gamma\left(v+a\right)}{\Gamma\left(v\right)}\nu^{-a} = 1 \tag{2.56}$$

and we limit ourselves to the first order of λ to write $\left(1-\lambda r^2\right)^{\frac{\mu}{2}} = \exp\left(-\frac{\Omega r^2}{2}\right)$ with $\Omega = \left(\frac{m^2\omega^2}{\hbar^2}+\frac{e^2B^2}{4\hbar^2c^2}\right)^{1/2}$. We thus obtain directly the position space eigenfunction of the usual KGO (without deformation):

$$\Psi\left(r,\varphi\right) = \sqrt{\frac{n!\Omega^{l+1}}{\pi\Gamma\left(n+l+1\right)}} e^{il\varphi}\exp\left(-\frac{\Omega r^2}{2}\right) r^l L_n^l\left(\Omega r^2\right) \tag{2.57}$$

An interesting characteristic of the spectrum appears when computing the energy levels spacing; indeed we find the limit:

$$\lim_{n\to\infty}\Delta E_{n,l} = 2\hbar c\sqrt{\lambda} \tag{2.58}$$

This expression shows that, without the effects of the deformed algebra, the energy spectrum of the KG oscillator under a strong magnetic field becomes almost continuous for large values of n. In contrast, this continuous feature of the spectrum disappears and it reduces to a bound spectrum in the deformed case. This asymptotic behavior is described by eq.(2.58) where the spacing is proportional to the deformation parameter λ.

In order to get an upper bound for this λ parameter, we use the $s-$states of the energies from eq.(2.42) and we expand it up to the first order in λ:

$$E_{n,0} = E_{n,0}^{\lambda=0} + \frac{\hbar^2 c^2}{2E_{n,0}^{\lambda=0}} \left[(2n+1)^2 - \frac{(2n+1)\,\omega}{\sqrt{\omega^2 + \widetilde{\omega}^2}} \right] \lambda \tag{2.59}$$

with:

$$E_{n,0}^{\lambda=0} = \sqrt{m^2 c^4 - 2\omega\hbar m c^2 + 2\,(2n+1)\,\hbar m c^2 \sqrt{\omega^2 + \widetilde{\omega}^2}} \tag{2.60}$$

These two relations show that the deviation of the n-th energy level caused by the modified commutation relations (2.1), is provided by:

$$\frac{\Delta E_{n,0}^{\lambda}}{\hbar\omega} = \frac{\hbar c^2}{2\omega E_{n,0}^{\lambda=0}} \left[(2n+1)^2 - \frac{(2n+1)\,\omega}{\sqrt{\omega^2 + \widetilde{\omega}^2}} \right] \lambda \tag{2.61}$$

We use the experimental results of the electron cyclotron motion in a Penning trap. Here, the cyclotron frequency of an electron trapped in a magnetic field of strength B is $\omega_c = eB/m_e$ (without deformation), so we have $m_e \hbar\omega_c = e\hbar B = 10^{-52} kg^2 m^2 s^{-2}$ for a magnetic field of strength $B = 6T$. If we assume that only the deviations of the scale of $\hbar\omega_c$ can be detected at the level $n = 10^{10}$ and that $\Delta E_n < \hbar\omega_c$ (no perturbation of the n-th energy level is observed) [54], we can write the following constraint:

$$\lambda < 3.36 \times 10^{-4} m^{-2} \tag{2.62}$$

This leads to the following upper bound of the minimal uncertainty in momentum $\Delta P_{\min} = \hbar\sqrt{\lambda} < 2 \times 0^{-36} kgms^{-1}$; it is similar to that obtained in [55].

For the non-relativistic limit, by setting $E = mc^2 + E_{nr}$ with the assumption that $mc^2 \gg E_{nr}$, we write the spectrum of the non-relativistic KGO in the deformed AdS space as:

$$E_{nr} = (2n+l+1)\,\hbar\sqrt{\left(\omega - \frac{\lambda\hbar}{2m}\right)^2 + \widetilde{\omega}^2} + \frac{\lambda\hbar^2}{2m}\left(4n\,(n+l+1) + 2l + 1\right) - \hbar\widetilde{\omega}l - \hbar\omega \tag{2.63}$$

2.5 Dirac oscillator in magnetic field

The Dirac oscillator in a commutative space is defined by the following substitution[44][45][56][57][58]

$$\left[c\widehat{\boldsymbol{\alpha}}.(\mathbf{p}+im\omega\widehat{\boldsymbol{\beta}}\mathbf{r})+\widehat{\boldsymbol{\beta}}mc^{2}\right]\Psi\left(\mathbf{r}\right)=E\Psi\left(\mathbf{r}\right), \tag{2.64}$$

where $\widehat{\boldsymbol{\alpha}}$ and $\widehat{\boldsymbol{\beta}}$ are the usual Dirac matrices, given by

$$\widehat{\boldsymbol{\alpha}}=\begin{pmatrix} 0 & \widehat{\boldsymbol{\sigma}} \\ \widehat{\boldsymbol{\sigma}} & 0 \end{pmatrix},\ \widehat{\boldsymbol{\beta}}=\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{2.65}$$

$\widehat{\boldsymbol{\sigma}}$ designates the Pauli matrices and m and ω, respectively, are the oscillator's mass and frequency .

We may assume that the four-component spinor Ψ is of the form

$$\Psi\left(\mathbf{r}\right)=\begin{pmatrix} \psi_{a}\left(\mathbf{r}\right) \\ \psi_{b}\left(\mathbf{r}\right) \end{pmatrix} \tag{2.66}$$

On substitution of Ψ as given above into (2.64)we get the following equations for the two-component spinors $\psi_{a}\left(\mathbf{r}\right)$ and $\psi_{b}\left(\mathbf{r}\right)$:

$$\begin{aligned} c\widehat{\boldsymbol{\sigma}}.(\mathbf{p}-\frac{e}{c}\mathbf{A}+im\omega\mathbf{r})\psi_{b}\left(\mathbf{r}\right)&=\left(E-mc^{2}\right)\psi_{a}\left(\mathbf{r}\right)\\ c\widehat{\boldsymbol{\sigma}}.(\mathbf{p}-\frac{e}{c}\mathbf{A}-im\omega\mathbf{r})\psi_{a}\left(\mathbf{r}\right)&=\left(E+mc^{2}\right)\psi_{b}\left(\mathbf{r}\right) \end{aligned} \tag{2.67}$$

Here $\mathbf{A}$ takes the definition as in (2.19)

By taking account of the definition of the Anti-de Sitter algebra (2.6)-(2.7) , the two dimensional deformed stationary Dirac oscillator equation can be put then,

$$\begin{aligned} c\widehat{\boldsymbol{\sigma}}.\mathbf{p}^{+}\psi_{b}\left(\mathbf{r}\right)&=\left(E-mc^{2}\right)\psi_{a}\left(\mathbf{r}\right)\\ c\widehat{\boldsymbol{\sigma}}.\mathbf{p}^{-}\psi_{a}\left(\mathbf{r}\right)&=\left(E+mc^{2}\right)\psi_{b}\left(\mathbf{r}\right) \end{aligned} \tag{2.68}$$

where we use the same definitions for $\mathbf{p}^{+}$and $\mathbf{p}^{-}$as in eq(2.21)

These two equations can be used to eliminate $\psi_{b}\left(\mathbf{r}\right)$ in favour of $\psi_{a}\left(\mathbf{r}\right)$, so that one can have

$$c^{2}\left(\widehat{\boldsymbol{\sigma}}.\mathbf{p}^{+}\right)\left(\widehat{\boldsymbol{\sigma}}.\mathbf{p}^{-}\right)\psi_{a}\left(\mathbf{r}\right)=\left(E^{2}-m^{2}c^{4}\right)\psi_{a}\left(\mathbf{r}\right) \tag{2.69}$$

According to the following relations:

$$\left(\widehat{\boldsymbol{\sigma}}.\mathbf{A}\right)\left(\widehat{\boldsymbol{\sigma}}.\mathbf{B}\right)=\mathbf{A}.\mathbf{B}+i\widehat{\boldsymbol{\sigma}}.\left(\mathbf{A}\times\mathbf{B}\right) \tag{2.70}$$

The equation (2.69) can be written as

$$c^2\left(\left(\mathbf{p}^{+}.\mathbf{p}^{-}\right)+i\widehat{\boldsymbol{\sigma}}.(\mathbf{p}^{+}\times\mathbf{p}^{-})\right)\psi_a\left(\mathbf{r}\right)=\left(E^2-m^2c^4\right)\psi_a\left(\mathbf{r}\right) \tag{2.71}$$

after a simple calculation,we obtain;

$$\begin{aligned}&\left[\left(1-\lambda r^2\right)p^2+\left(\eta+\frac{eB}{2c}\left(2m\omega-\lambda\hbar\right)\sigma_z\right)\frac{r^2}{1-\lambda r^2}+i\hbar\lambda\mathbf{r}.\mathbf{p}\right.\\&\left.-\left(\frac{eB}{c}+\left(2m\omega+\lambda\hbar\right)\sigma_z\right)L_z-\frac{eB\hbar}{c}\sigma_z-\varepsilon\right]\psi_a\left(\mathbf{r}\right)=0\end{aligned} \tag{2.72}$$

where

$$\eta=m^2\omega^2+\frac{e^2B^2}{4c^2}-\lambda m\omega\hbar \text{ and } \varepsilon=\frac{E^2-m^2c^4}{c^2}+2m\omega\hbar. \tag{2.73}$$

To solve the equation.(2.72), we use the following ansatz $\psi_a\left(\mathbf{r}\right)=e^{il\varphi}R_{nl}(r)\chi_\tau$, where n is the radial quantum number, l and $\tau=\pm1$ are, respectively, the eigenvalues of angular momentum and spin operators, and $\chi_{+1}^T=(1,0)\,,\,\chi_{-1}^T=(0,1)$ are the spin functions.

Using the polar coordinates of the position and momentum operators, we obtain the following differential equation for the radial part of the wave function:

$$\left[\left(\sqrt{1-\lambda r^2}\frac{d}{dr}\right)^2-\frac{l^2\left(1-\lambda r^2\right)}{r^2}-\frac{\eta^\tau r^2}{\hbar^2\left(1-\lambda r^2\right)}+\epsilon^\tau\right]R_{nl}(r)=0 \tag{2.74}$$

with

$$\eta^\tau=\eta+\frac{eB}{2c}\left(2m\omega-\lambda\hbar\right)\tau \text{ and } \epsilon^\tau=\frac{\varepsilon}{\hbar^2}+\frac{eB}{c\hbar}\tau+\frac{l}{\hbar}\left(\frac{eB}{c}+\left(2m\omega+\lambda\hbar\right)\tau\right) \tag{2.75}$$

Now to solve eq.(2.74), we use the transformation which is the same as in the previous section of KG equation,and following the similar procedure with the NU method,taking into account the conditions of eqs.(2.12) and (2.13), we obtain:

$$\begin{aligned}\Lambda_n &= k_2-\frac{l}{2}=n\left(n+\mu^\tau+l+\frac{1}{2}\right),n=0,1,2,...\\ \text{With } \mu^\tau &= \frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\eta^\tau}{\hbar^2\lambda^2}}\end{aligned} \tag{2.76}$$

Hence, we found the expressions of the energy eigenvalues as:

$$E_{n,l,\tau} = \pm mc^2 \left[1 - \frac{2\omega\hbar}{mc^2}(l\tau+1) + \frac{2\hbar}{mc^2}\left\{(2n+l+1)\sqrt{\left(\omega - \frac{\lambda\hbar}{2m}\right)^2 + \widetilde{\omega}^2 + 2\widetilde{\omega}\left(\omega - \frac{\lambda\hbar}{2m}\right)\tau} + \frac{\lambda\hbar}{2m}\left(4n(n+l+1) + (2-\tau)l + 1\right) - \widetilde{\omega}(l+\tau)\right\}\right]^{\frac{1}{2}} \quad (2.77)$$

As the KG energy spectrum in eq(2.42),we can see that the energy spectrum of the system contains corrections associated with the deformation; the first one comes with the oscillator term and an additional one which increases with the deformation parameter λ,the last one related to $\widetilde{\omega}$ (i.e.B) and τ which presents the value of the spin (± 1). In addition, according to the n^2 dependence of the energy levels, which corresponds to a confinement at the high energy area, in our case, the boundaries of the well are placed at $\pm\pi/2\sqrt{\lambda}$

We can test the obtained expression of the energy by different ways.For instant, if we put $\tau \to 0$,we obtain the KG deformed enegy spectrum expression as it mentioned in (2.42).More than this, when $\lambda = 0$ (i.e ; without deformation), we get the DO in a magnetic field in the literature[59], also if we put $\widetilde{\omega} = 0$ ($B = 0$) in the absence of the magnetic field, we obtain the DO in the commutative space[56]

By a straightforward calculation, it is easy to solve this equation exactly in the same way as in the case of KG and to obtain the relativistic spinor wave functions in terms of Jacobi polynomial as

$$\Psi_{n,m_l}(r,\varphi) = C^{`} 2^{\frac{l}{2}} e^{il\varphi}\left(\lambda r^2\right)^{\frac{l}{2}} \times$$

$$\left[\begin{array}{c}\begin{pmatrix}1\\0\\0\\\Lambda\left(r\right)e^{i\varphi}\end{pmatrix}\left(1-\lambda r^{2}\right)^{\frac{\mu^{+}}{2}}P_{n}^{\left(l,\mu^{+}-\frac{1}{2}\right)}\left(1-2\lambda r^{2}\right)\\+\begin{pmatrix}0\\1\\\Gamma\left(r\right)e^{-i\varphi}\\0\end{pmatrix}\left(1-\lambda r^{2}\right)^{\frac{\mu^{-}}{2}}P_{n}^{\left(l,\mu^{-}-\frac{1}{2}\right)}\left(1-2\lambda r^{2}\right)-\\\begin{pmatrix}0\\0\\0\\\Pi^{+}\left(r\right)e^{i\varphi}\end{pmatrix}\left(1-\lambda r^{2}\right)^{\frac{\mu^{+}}{2}}P_{n-1}^{\left(l+1,\mu^{+}+\frac{1}{2}\right)}\left(1-2\lambda r^{2}\right)\\+\begin{pmatrix}0\\0\\\Pi^{-}\left(r\right)e^{-i\varphi}\\0\end{pmatrix}\left(1-\lambda r^{2}\right)^{\frac{\mu^{-}}{2}}P_{n-1}^{\left(l+1,\mu^{-}+\frac{1}{2}\right)}\left(1-2\lambda r^{2}\right)\end{array}\right] \tag{2.78}$$

with

$$\Gamma\left(r\right)=\frac{i\hbar}{E+mc^{2}}\left(\frac{2l\sqrt{1-\lambda r^{2}}}{r}-\left(\frac{eB}{2c}+m\omega-\lambda\mu^{-}\right)\frac{r}{\sqrt{1-\lambda r^{2}}}\right). \tag{2.79}$$

$$\Lambda\left(r\right)=\frac{-i\hbar}{E+mc^{2}}\left(\frac{eB}{2c}+m\omega-\lambda\mu^{+}\right)\frac{r}{\sqrt{1-\lambda r^{2}}}. \tag{2.80}$$

$$\Pi^{\mp}\left(r\right)=\frac{2i\hbar\lambda}{E+mc^{2}}\left(n+l+\mu^{\mp}+\frac{1}{2}\right)r\sqrt{1-\lambda r^{2}}. \tag{2.81}$$

where the following properties Jacobi polynomial were used

$$\frac{dP_{n}^{(a,b)}\left(y\right)}{dy}=\frac{1}{2}\left(n+a+b+1\right)P_{n-1}^{(a+1,b+1)}\left(y\right) \tag{2.82}$$

2.5.1 Special case for graphene

We consider the (2+1)-dimensional massless Dirac equation that describes the motion of electrons with Fermi velocity $V_F=(1.12\pm0.02)\,10^{6}\ ms^{-1}$ in quantum theory of graphene, the Hamiltonian is given by

$$(\widehat{\boldsymbol{\alpha}}.\mathbf{p})\,\Psi(\mathbf{r})=\frac{E}{V_F}\Psi(\mathbf{r}) \tag{2.83}$$

By taking into consideration the computation as the Dirac oscillator in a magnetic field in AdS, as it mentioned in the above section and we use the limit $m\rightarrow 0$;hence,the energy eigenvalues are found as

$$\begin{aligned} E^{\lambda}_{n,l,\tau} &= \pm\frac{\hbar V_F}{l_B}\Big[(2n+l+1)\sqrt{1-2\lambda l_B^2\tau+\lambda^2 l_B^4} \\ &\quad +\lambda l_B^2\left(4n\left(n+l+1\right)+\left(2-\tau\right)l+1\right)-\left(l+\tau\right)\Big]^{\frac{1}{2}} \end{aligned} \tag{2.84}$$

And here by replacing $\tau=\pm 1$,we get:

$$E^{\lambda}_{n,l,+1} = \pm\frac{\hbar V_F}{l_B}\left[2n\left(1+\lambda l_B^2\left(2n+2l+1\right)\right)\right]^{\frac{1}{2}} \tag{2.85}$$

$$E^{\lambda}_{n,l,-1} = \pm\frac{\hbar V_F}{l_B}\left[2\left(n+1\right)\left(1+\lambda l_B^2\left(2n+2l+1\right)\right)\right]^{\frac{1}{2}} \tag{2.86}$$

where $l_B=\sqrt{\frac{\hbar}{eB}}$ is the fundamental length scale in the presence of a magnetic field

In terms of the variables r and φ, we can now write the general form of the wave function Ψ as follows:

$$\Psi_n\left(r,\varphi\right)=C_n 2^{\frac{l}{2}}e^{il\varphi}\left(1-\lambda r^2\right)^{\frac{\mu}{2}}\left(\lambda r^2\right)^{\frac{l}{2}}P_n^{\left(l,\mu-\frac{1}{2}\right)}\left(1-2\lambda r^2\right) \tag{2.87}$$

where C_n is a normalization constant.

2.6 DKP oscillator in a magnetic field

The free DKP equation of massive scalar and vector particles m can be written as follows[46][60][61][62][63][6

[65][66]

$$\left[c\boldsymbol{\beta}\cdot\mathbf{p}+mc^2\right]\Psi\left(\mathbf{r},t\right)=i\hbar\beta^0\partial_0\Psi\left(\mathbf{r},t\right) \tag{2.88}$$

where $\boldsymbol{\beta}$ and β^0 are the DKP matrices[64][65].

We write the 2D DKP oscillator in a same way as the Dirac oscillator[56] and so, we introduce the non-minimal substitution [60]:

$$\mathbf{p}\rightarrow\mathbf{p}-im\omega\eta^0\mathbf{r} \tag{2.89}$$

where ω is the frequency of the oscillator and η^0 is defined by the relation $\eta^0 = 2\left(\beta^0\right)^2 - 1$ (note that $\left(\eta^0\right)^2 = 1$).

Using the relation (2.89) in eq.(2.88), we get the equation of the DKP oscillator and we add a vector potential $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$:

$$\left[c\boldsymbol{\beta}\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}-im\omega\eta^0\mathbf{r}\right)+mc^2\right]\Psi = i\hbar\beta^0\left(\frac{\partial\Psi}{\partial t}\right) \tag{2.90}$$

By taking the definition of the AdS algebra (2.1), we write the 2D deformed stationary DKP oscillator equation for the stationary solutions ($\Psi\left(\mathbf{r},t\right) = e^{-iEt/\hbar}\tilde{\Psi}\left(\mathbf{r}\right)$):

$$\left[c\boldsymbol{\beta}\cdot\left(\sqrt{1-\lambda r^2}\mathbf{p}-\frac{e}{c}\mathbf{B}\times\frac{\mathbf{r}}{\sqrt{1-\lambda r^2}}-im\omega\eta^0\frac{\mathbf{r}}{\sqrt{1-\lambda r^2}}\right)+mc^2\right]\tilde{\Psi} = E\beta^0\tilde{\Psi} \tag{2.91}$$

2.6.1 Scalar particle case

In the case of a scalar particle (spin 0), the wave function is a vector with five components:

$$\tilde{\Psi}\left(\mathbf{r}\right) = \begin{pmatrix}\Phi\\ i\boldsymbol{\psi}\end{pmatrix} \text{ with } \Phi \equiv \begin{pmatrix}\phi\\ \chi\end{pmatrix} \text{ and } \boldsymbol{\psi} \equiv \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} \tag{2.92}$$

When we substitute eq.(2.92) into eq.(2.91), we obtain the following coupled system:

$$mc^2\phi = E\chi + ic\mathbf{p}^+\cdot\boldsymbol{\psi} \tag{2.93}$$

$$mc^2\boldsymbol{\psi} = ic\mathbf{p}^-\phi \tag{2.94}$$

$$mc^2\chi = E\phi \tag{2.95}$$

where the expressions of $\mathbf{p}^+$ and $\mathbf{p}^-$ are given in eq.(2.21).

At this stage, the system is uncoupled in favour of ϕ and so, it can be converted directly to the same differential equation of the 2D deformed KG case (2.22):

$$\left[\left(1-\lambda r^2\right)p^2 + \frac{\eta r^2}{1-\lambda r^2} + i\hbar\lambda\mathbf{r}.\mathbf{p} - \frac{eB}{c}L_z - \varepsilon\right]\phi = 0 \tag{2.96}$$

where η and ε have the same definitions given in the above section 2 (2.23).

Following the same reasoning as in the KG case and so, according to the condition (2.31), the exact solution of the scalar DKP is given by ($C^{\backslash}$ is a normalization constant and

μ is given in (2.33)):

$$\phi_n(r,\varphi) = C^{\backprime} 2^{\frac{l}{2}} e^{il\varphi} \left(1-\lambda r^2\right)^{\frac{\mu}{2}} \left(\lambda r^2\right)^{\frac{l}{2}} P_n^{\left(l,\mu-\frac{1}{2}\right)} \left(1-2\lambda r^2\right) \tag{2.97}$$

and the corresponding energy spectrum is giving with:

$$E_{n,l} = \pm mc^2 \left[1 - \frac{2\omega\hbar}{mc^2} + \frac{2\hbar}{mc^2}\left\{(2n+l+1)\sqrt{\left(\omega-\frac{\lambda\hbar}{2m}\right)^2 + \widetilde{\omega}^2} + \frac{\lambda\hbar}{2m}\left(4n\left(n+l+1\right)+2l+1\right) - \widetilde{\omega} l\right\}\right]^{\frac{1}{2}} \tag{2.98}$$

The other components are simple to assess, and the final expressions of $\tilde{\Psi}\left(\mathbf{r}\right)$ are as follows:

$$\tilde{\Psi}_{n,l}(r,\varphi) = C^{\backprime} 2^{\frac{l}{2}} e^{il\varphi} \left(1-\lambda r^2\right)^{\frac{\mu}{2}} \left(\lambda r^2\right)^{\frac{l}{2}} \times$$

$$\left[\begin{pmatrix} 1 \\ \frac{E}{mc^2} \\ M\left(r\right) \\ N\left(r\right) \\ 0 \end{pmatrix} P_n^{\left(l,\mu-\frac{1}{2}\right)}\left(1-2\lambda r^2\right) - \begin{pmatrix} 0 \\ 0 \\ \Lambda\left(r\right) \\ 0 \\ 0 \end{pmatrix} P_{n-1}^{\left(l+1,\mu+\frac{1}{2}\right)}\left(1-2\lambda r^2\right)\right] \tag{2.99}$$

with:

$$M\left(r\right) = \frac{\hbar l}{mc}\frac{\sqrt{1-\lambda r^2}}{r} + \left(\omega - \frac{\lambda\hbar\mu}{m}\right)\frac{r}{c\sqrt{1-\lambda r^2}} \tag{2.100}$$

$$N\left(r\right) = \frac{i}{mc}\left(\frac{\hbar l\sqrt{1-\lambda r^2}}{r} - \frac{eBr}{2c\sqrt{1-\lambda r^2}}\right) \tag{2.101}$$

$$\Lambda\left(r\right) = \frac{2\lambda\hbar}{m}\left(n+l+\mu+\frac{1}{2}\right) r\sqrt{1-\lambda r^2} \tag{2.102}$$

Before ending this section, we must calculate the normalization constant $C^{\backprime}$ using the following normalization condition:

$$\int \frac{rdr}{\sqrt{1-\lambda r^2}} \overline{\Psi}(r,\varphi)\beta^0\Psi(r,\varphi) = 1 \quad \text{with } \overline{\Psi} = \Psi^{+}\eta^0 \tag{2.103}$$

We use the components of the spinor Ψ to write it as:

$$\int \frac{rdr}{\sqrt{1-\lambda r^2}} \left[\phi^{*}\varphi + \varphi^{*}\phi\right] = 1 \tag{2.104}$$

Then with the help of the Jacobi polynomial orthogonality relation (2.54), we obtain:

$$C^{\backprime} = \sqrt{\frac{\lambda mc^2}{2^{l+1}\pi E}\frac{n!\left(2n+\mu+l+\frac{1}{2}\right)\Gamma\left(n+\mu+l+\frac{1}{2}\right)}{\Gamma\left(n+\mu+\frac{1}{2}\right)\Gamma\left(n+l+1\right)}} \tag{2.105}$$

Thus we obtain the final form of the solutions $\tilde{\Psi}\left(\mathbf{r}\right)$.

2.6.2 Vector particle case

The wave function of the spin 1 particle is a vector with ten components noted by $\tilde{\Psi}(\mathbf{r})^T = (i\varphi, \mathbf{A}(\mathbf{r}), \mathbf{B}(\mathbf{r}), \mathbf{C}(\mathbf{r}))$, with A_i, B_i and C_i $(i = 1,2,3)$ being respectively the components of the vectors $\mathbf{A}(\mathbf{r}), \mathbf{B}(\mathbf{r})$ and $\mathbf{C}(\mathbf{r})$. Puttig this form in eq.(2.91) gives us the following system:

$$mc^2\varphi = -c\,\mathbf{p}^- \cdot \mathbf{B} \tag{2.106}$$

$$mc^2\mathbf{A} = E\,\mathbf{B} - c\,\mathbf{p}^+ \times \mathbf{C} \tag{2.107}$$

$$mc^2\mathbf{B} = E\,\mathbf{A} + c\,\mathbf{p}^+\varphi \tag{2.108}$$

$$mc^2\mathbf{C} = -c\,\mathbf{p}^- \times \mathbf{A} \tag{2.109}$$

To decouple this system, we eliminate φ, $\mathbf{B}$ and $\mathbf{C}$ in terms of $\mathbf{A}$ and we get:

$$\left(E^2 - m^2c^4\right)\mathbf{A} = c^2\mathbf{p}^+\left(\mathbf{p}^- \cdot \mathbf{A}\right) - c^2\mathbf{p}^+ \times \left(\mathbf{p}^- \times \mathbf{A}\right) - \frac{1}{m^2}\mathbf{p}^+\left[\mathbf{p}^- \cdot \left[\mathbf{p}^+ \times \left(\mathbf{p}^- \times \mathbf{A}\right)\right]\right] \tag{2.110}$$

and we rewrite it in the following form:

$$\left(E^2 - m^2c^4\right)\mathbf{A} = c^2\left[\left(\mathbf{p}^+ \cdot \mathbf{p}^-\right)\mathbf{A} - \left(\mathbf{p}^+ \times \mathbf{p}^-\right) \times \mathbf{A}\right] - \frac{1}{m^2}\mathbf{p}^+\left[\mathbf{p}^- \cdot \left[\mathbf{p}^+ \times \left(\mathbf{p}^- \times \mathbf{A}\right)\right]\right] \tag{2.111}$$

A direct calculation of the first two terms in eq.(2.111) gives:

$$\begin{aligned}\left(\mathbf{p}^+ \cdot \mathbf{p}^-\right)\mathbf{A} = \Bigg[&\left(m^2\omega^2 + \frac{e^2B^2}{4c^2} - m\omega\hbar\lambda\right)\frac{r^2}{1-\lambda r^2} + \left(1-\lambda r^2\right)p^2 + i\hbar\lambda\mathbf{r}\cdot\mathbf{p} - \\ &\frac{eB}{c}L_z - 2m\omega\hbar\Bigg]\mathbf{A}\end{aligned} \tag{2.112}$$

$$\left(\mathbf{p}^+ \times \mathbf{p}^-\right) \times \mathbf{A} = \left[\frac{eB}{c}\left(\lambda - \frac{2m\omega}{\hbar}\right)\frac{r^2}{1-\lambda r^2} + 2\left(\frac{eB}{c} + \left(\lambda + \frac{2m\omega}{\hbar}\right)L_z\right)\right]S_z\mathbf{A} \tag{2.113}$$

Inserting these results into eq.(2.111), we get:

$$\begin{aligned}\varepsilon^{\backprime}\mathbf{A} = \Bigg[&\left(1-\lambda r^2\right)p^2 + \left(m^2\omega^2 + \frac{e^2B^2}{4c^2} - \frac{eB}{c}\left(\lambda - \frac{2m\omega}{\hbar}\right)S_z - m\omega\hbar\lambda\right)\frac{r^2}{1-\lambda r^2} + \\ &i\hbar\lambda\mathbf{r}\cdot\mathbf{p} - \left(\frac{eB}{c} + 2\left(\lambda + \frac{2m\omega}{\hbar}\right)S_z\right)L_z\Bigg]\mathbf{A} - \frac{1}{(mc)^2}\mathbf{p}^+\left[\mathbf{p}^- \cdot \left[\mathbf{p}^+ \times \left(\mathbf{p}^- \times \mathbf{A}\right)\right]\right]\end{aligned} \tag{2.114}$$

with:

$$\varepsilon^{\backprime} = \frac{E^2 - m^2c^4}{c^2} + 2m\omega\hbar + \frac{2eB}{c}S_z \tag{2.115}$$

and L_z and S_z are the z-components of the orbital angular momentum and the spinor respectively.

To our knowledge, there is no analytical method to solve equations like eq.(2.114) but that does not prevent us from studying its nonrelativistic limit; this is perfomed by considering the last term as negligible since it is of the order m^{-3}.

Using this approximation, eq.(2.114) becomes similar to both eq.(2.22) and eq.(2.91) corresponding to the KG case and to the scalar DKP particle respectively; we find:

$$\begin{aligned}\varepsilon^{\backprime}\mathbf{A} = \Bigg[&\left(1-\lambda r^2\right)p^2 + \left(m^2\omega^2 + \frac{e^2B^2}{4c^2} - \frac{eB}{c}\left(\lambda - \frac{2m\omega}{\hbar}\right)S_z - \lambda\hbar m\omega\right)\frac{r^2}{1-\lambda r^2} + \\ &i\hbar\lambda\mathbf{r}\cdot\mathbf{p} - \left(\frac{eB}{c} + 2\left(\lambda + \frac{2m\omega}{\hbar}\right)S_z\right)L_z\Bigg]\mathbf{A}\end{aligned} \tag{2.116}$$

Comparing this equation to both eq.(2.22) and eq.(2.96), we find four additional terms. Two of them appear in the harmonic part ($\propto r^2$); the pure spin term $2m\omega S_z/\hbar$ and the interaction term $2m\omega eBS_z/\hbar c$. The other two are of spin-orbit type; the usual one $4m\omega S_zL_z/\hbar$ and the additional $2\lambda S_zL_z$ coming from the influence of the space deformation. As it should be, all these terms are due to the presence of the spin in this vectorial case.

We follow the same procedure used in the precedent sections, to obtain the energies and we get:

$$\begin{aligned}E^{nr} = &\left(2n+l+1\right)\hbar\sqrt{\left(\omega - \frac{\lambda\hbar}{2m}\right)^2 + \widetilde{\omega}^2 - \frac{4}{\hbar}\widetilde{\omega}\left(\omega - \frac{\lambda\hbar}{2m}\right)S_z} - \left(\hbar\omega + \widetilde{\omega}S_z\right) + \\ &\frac{\lambda\hbar^2}{2m}\left(4n\left(n+l+1\right)+2l+1\right) - \left[\widetilde{\omega} + \frac{2}{\hbar}\left(\omega + \frac{\lambda\hbar}{2m}\right)S_z\right]l\end{aligned} \tag{2.117}$$

The zero value of S_z ($m_s = 0$) gives us the same spectrum of the scalar particle (eq.2.42). So, when considering the nonzero eigenvalues of S_z ($m_s = \pm 1$), we write the final expression of the non-relativistic energy spectrum as follows:

$$\begin{aligned}E^{nr} = &\left(2n+l+1\right)\hbar\sqrt{\left(\omega - \frac{\lambda\hbar}{2m}\right)^2 + \widetilde{\omega}^2 \mp 4\widetilde{\omega}\left(\omega - \frac{\lambda\hbar}{2m}\right)} - \hbar\left(\omega \pm 2\widetilde{\omega}\right) + \\ &\frac{\lambda\hbar^2}{2m}\left(4n\left(n+l+1\right)+2l+1\right) - \left[\widetilde{\omega} \pm 2\left(\omega + \frac{\lambda\hbar}{2m}\right)\right]l\end{aligned} \tag{2.118}$$

The result explicitly shows the contributions of all the terms in eq.(2.116) and especially those due to the presence of both the spin and the deformation, as well as that of the additional spin-orbit term $2\lambda S_z L_z$ which can be interpreted as due to the interaction between the two. The same result is found in the case of Dirac oscillator with EUP [67][68].

2.7 Thermodynamic properties of KG and DKP equations

In statistical mechanics, the partition function Z is an important quantity that encodes the statistical properties of a system in thermodynamic equilibrium. It is a function of temperature and other parameters, such as the volume enclosing a gas. Most of the thermodynamic variables of the system, such as the total energy, free energy, entropy, and pressure, can be expressed in terms of the partition function or its derivatives.

Now we focus on the thermodynamic properties of the system.The partition function at finite temperature T is:

$$Z = \sum_{n=0}^{\infty} e^{-\frac{E_n}{k_B T}} = \sum_{n=0}^{\infty} \exp\left(-\frac{mc^2}{k_B T}\sqrt{a_1 + a_2 n + a_3 n^2}\right) \tag{2.119}$$

Here k_B is the Boltzmann constant and the expressions of the other parameters obtain from the energy spectrum eq(2.42):

$$a_1 = 1 + \frac{2\hbar}{mc^2}\left((l+1)\sqrt{\left(\omega - \frac{\lambda\hbar}{2m}\right)^2 + \widetilde{\omega}^2} + \frac{\lambda\hbar}{2m}(2l+1) - \widetilde{\omega} l - \omega\right)$$

$$a_2 = \frac{4\hbar}{mc^2}\left(\sqrt{\left(\omega - \frac{\lambda\hbar}{2m}\right)^2 + \widetilde{\omega}^2} + \frac{\lambda\hbar}{m}(l+1)\right) \text{ and } a_3 = \frac{4\lambda\hbar^2}{m^2c^2} \tag{2.120}$$

In order to evaluate this function, we use the Euler-MacLaurin formula:

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int f(x)\, dx - \sum_{p=1}^{\infty} \frac{1}{(2p)!} B_{2p} f^{(2p-1)}(0) \tag{2.121}$$

B_{2p} are the Bernoulli numbers, $f^{(2p-1)}$ is the derivative of order $(2p-1)$ and the integral term is given by:

$$I = \frac{2a_1}{\sqrt{a_2^2 - 4a_1a_3}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \left(\frac{4a_1a_3}{a_2^2 - 4a_1a_3}\right)^n \times \left[\frac{\Gamma(2n+2)}{\chi^{2n+2}} - \frac{e^{-\chi}}{(2n+2)} \Phi(1, 2n+2, \chi)\right] \tag{2.122}$$

where we have used $\chi = \frac{mc^2}{k_BT}\sqrt{a_1}$, the new variable $y = \sqrt{1 + \frac{a_2}{a_1}n + \frac{a_3}{a_1}n^2}$and the power series of the square root of the integral:

$$I = \frac{2a_1}{\sqrt{a_2^2 - 4a_1a_3}} \int_1^{+\infty} \exp\left(-\chi y\right) \left(1 + \frac{4a_1a_3}{a_2^2 - 4a_1a_3} y^2\right) ydy \tag{2.123}$$

At high temperatures, we can ignore the first and the third terms in eq.(2.121) and keep only the integral. Similarly, we neglect the $e^{-\chi}$ term beside the $\chi^{-(2n+2)}$ one in eq.(2.122). Therefore, the partition function becomes:

$$Z = \left(\frac{k_BT}{mc^2}\right)^2 \frac{2}{\sqrt{a_2^2 - 4a_1a_3}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \Gamma(2n+2)\sigma^n \tag{2.124}$$

with:

$$\sigma = \left(\frac{k_BT}{mc^2}\right)^2 \frac{4a_3}{\left(a_2^2 - 4a_1a_3\right)} \tag{2.125}$$

We restrict ourselves to the first order in λ, so we can reduce Z to the following simplified form:

$$Z \simeq \frac{\left(k_BT\right)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2 + \omega^2}} - \frac{3\left(k_BT\right)^4 \lambda}{2\hbar m^3c^4 \sqrt[3]{\widetilde{\omega}^2 + \omega^2}} \left(1 - \frac{mc^2\left(2\hbar\widetilde{\omega}l + mc^2 + \hbar\omega\right)}{6\left(k_BT\right)^2}\right) \tag{2.126}$$

We neglect the $\left(k_BT\right)^{-2}$ term in parentheses and we obtain the high-temperature expansion of the partition function:

$$Z \simeq \frac{\left(k_BT\right)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2 + \omega^2}} \left(1 - \frac{3\left(k_BT\right)^2 \lambda}{m^2c^2\left(\widetilde{\omega}^2 + \omega^2\right)}\right) \tag{2.127}$$

The first term is the usual partition function for a 2D scalar bosonic oscillator with a uniform magnetic field in the r-representation. The second term expresses the contribution that comes from the space deformation through the AdS algebra.

At this stage, we can evaluate all the thermodynamic properties of our system (free energy F, mean energy U, specific heat C and entropy S) using their definitions [37]:

$$F = -k_BT\ln Z, U = k_BT^2\frac{\partial \ln Z}{\partial T}, C = \frac{\partial U}{\partial T} \text{ and } S = -\frac{\partial F}{\partial T} \tag{2.128}$$

$$F = -k_B T \ln\left(\frac{(k_B T)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2+\omega^2}}\left(1-\theta\,(k_B T)^2\right)\right) \tag{2.129}$$

$$U = 4k_B T\left[1-\frac{1}{2\left(1-\theta\,(k_B T)^2\right)}\right] \tag{2.130}$$

$$C = 4k_B\left[1-\frac{1+\theta\,(k_B T)^2}{2\left(1-\theta\,(k_B T)^2\right)^2}\right] \tag{2.131}$$

$$S = k_B\left[\frac{2-4\theta\,(k_B T)^2}{1-\theta\,(k_B T)^2}+\ln\left(\frac{(k_B T)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2+\omega^2}}\left(1-\theta\,(k_B T)^2\right)\right)\right] \tag{2.132}$$

where we have used the parameter $\theta = \frac{3\lambda}{m^2c^2\left(\widetilde{\omega}^2+\omega^2\right)}$.

We can easily test these expressions in different ways. Using the limit $\widetilde{\omega} \to 0$ (i.e. $B \to 0$), we obtain the thermodynamic results of the deformed 2D scalar bosonic oscillator with AdS commutation relations. If we use the limit $\lambda \to 0$, we obtain the thermal properties of the ordinary 2D bosonic oscillator for both KG and scalar DKP particles in a uniform magnetic field.

We show in figures 2.1 to 2.4, the dependence of these thermodynamic properties with the temperature T for different values of the deformation parameter λ. We chose $\omega = B = 1$ andwe use the Hartree atomic units ($\hbar = c = k_B = m = 1$).

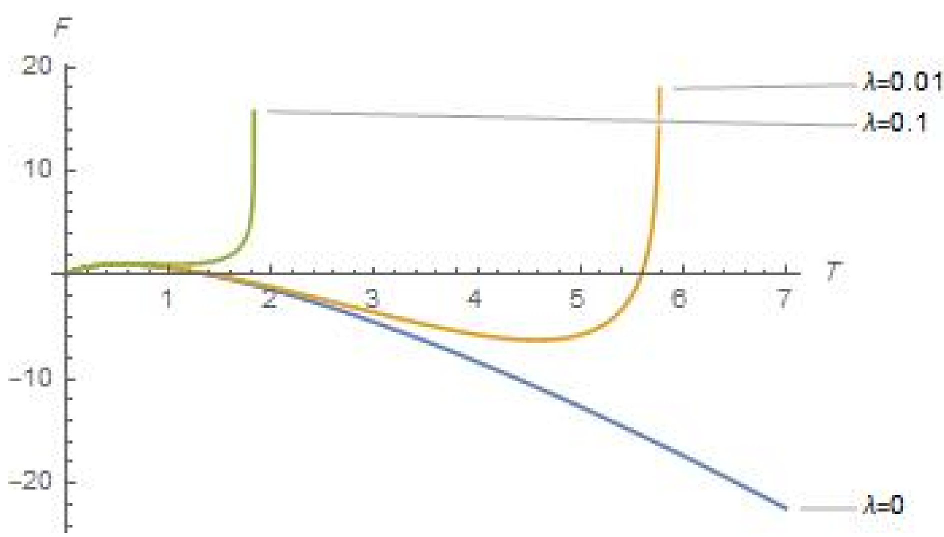
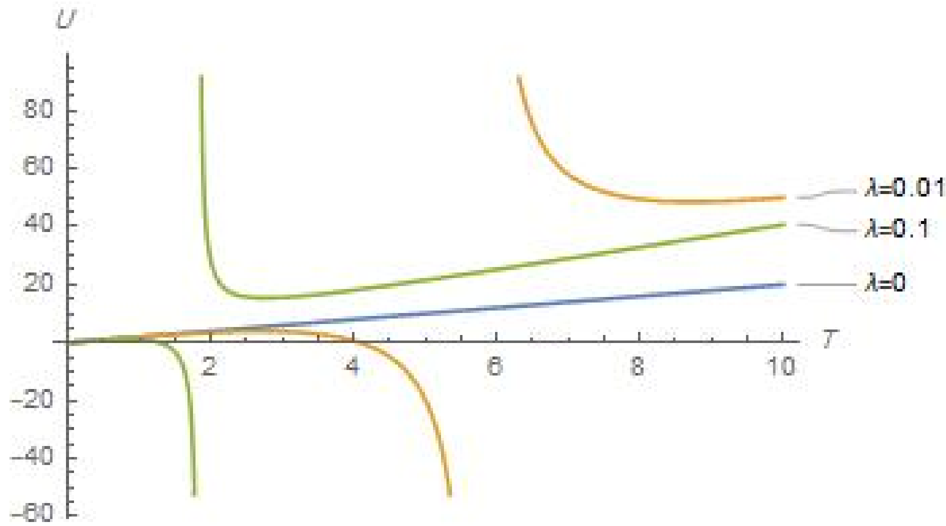
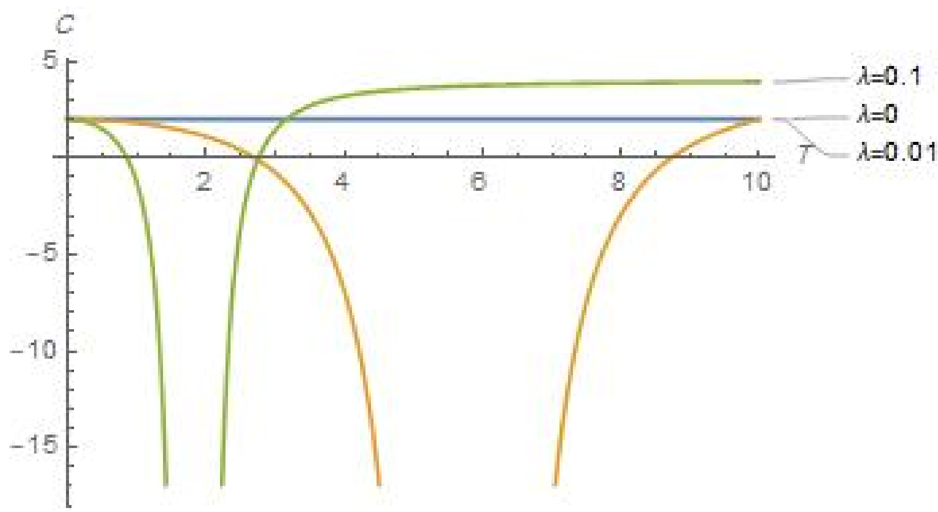


Figure.2.1 The free energy function F according to T for various values of the deformation parameter λ.

(2.133)



$Figure.2.2$ The mean energy U according to T for various values of the deformation parameter λ.



$Figure.2.3$ The specific heat function C according to T for various values of the deformation parameter λ.

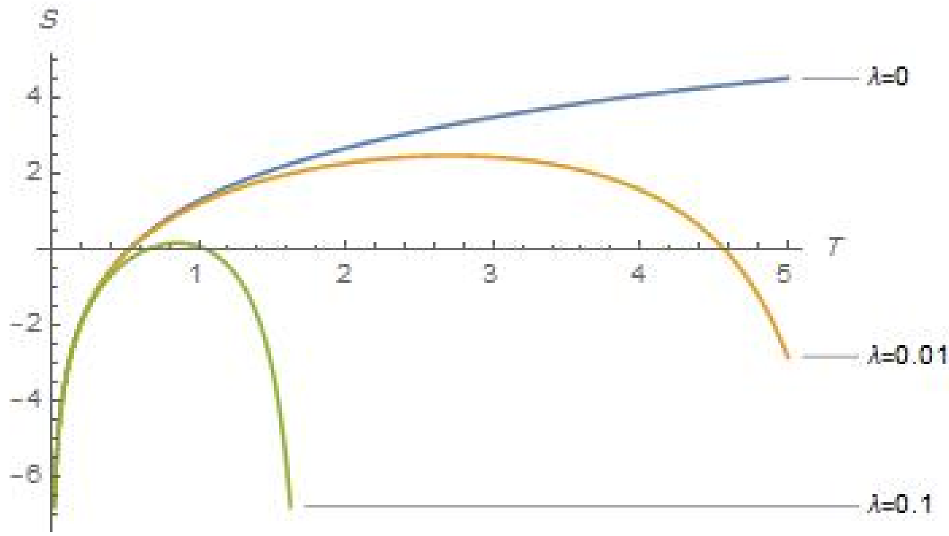


Figure.2.4 The entropy S according to T for various values of the deformation parameter λ.

The first remark that appears is that we have a critical point at $T_c = \sqrt{\frac{1+\alpha/4}{3\lambda}} = 1/\sqrt{3\lambda}$ (α is the fine structure constant) given the presence of the factor $1 - \theta\,(k_B T)^2$ in the denominator of U, C, and S (in eqs.(2.130), (2.131) and (2.132)). This radically changes the behavior of these thermodynamic properties as we will see in their graphical representations.

For the free energy, Fig.2.1 shows that it grows rapidly in the deformed case ($\lambda \neq 0$) unlike the normal case where it continues its decrease to infinity; this after a small increase followed by a decrease common to both cases. For a fixed value of T , the free energy increases with the deformation parameter λ.

Fig.2.2 shows that there is a discontinuity in the mean energy U at T_c where it decreases rapidly in the vicinity of T_c and then it grows slightly as in the ordinary case.

In Fig.2.3, we see that the specific heat C decreases until the discontinuity point T_c, then it increases. On the other hand, it is constant for $\lambda = 0$ (namely, $C = 2k_B$).

For the entropy function S, we see in Fig.2.4 that it grows to a maximum value then decreases to infinity and this contrasts with the ordinary case where it has a continuous growth with T

2.8 Thermodynamic properties of Dirac equation

We examine the thermodynamic properties of the Dirac oscillator in a magnetic field in AdS.The partition function at finite temperature T is:

$$Z=\sum_{n=0}^{\infty}e^{-\frac{E_n}{k_BT}}=\sum_{n=0}^{\infty}\exp\left(-\frac{mc^2}{k_BT}\sqrt{a_1+a_2n+a_3n^2}\right) \tag{2.134}$$

Here k_B is the Boltzmann constant and the expressions of the other parameters obtain from the energy spectrum eq(2.77):

$$\begin{aligned} a_1 &= 1+\frac{2\hbar}{mc^2}\left[(l+1)\sqrt{\left(\omega-\frac{\lambda\hbar}{2m}\right)^2+\widetilde{\omega}^2+2\widetilde{\omega}\left(\omega-\frac{\lambda\hbar}{2m}\right)\tau}\right. \\ &\left.+\frac{\lambda\hbar}{2m}\left((2-\tau)\,l+1\right)-\widetilde{\omega}\,(l+\tau)-\omega\,(l\tau+1)\right] \\ a_2 &= \frac{4\hbar}{mc^2}\left(\sqrt{\left(\omega-\frac{\lambda\hbar}{2m}\right)^2+\widetilde{\omega}^2+2\widetilde{\omega}\left(\omega-\frac{\lambda\hbar}{2m}\right)\tau}+\frac{\lambda\hbar}{m}(l+1)\right) \text{ and } a_3=\frac{4\lambda\hbar^2}{m^2c^2} \end{aligned} \tag{2.135}$$

We restrict ourselves to the first order in λ, so we can reduce Z to the following simplified form:

$$\begin{aligned} Z &\simeq \frac{(k_BT)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2+\omega^2++2\widetilde{\omega}\omega\tau}}-\frac{3\,(k_BT)^4\,\lambda}{2\hbar m^3c^4\sqrt[3]{\widetilde{\omega}^2+\omega^2++2\widetilde{\omega}\omega\tau}} \\ &\times\left(1-\frac{mc^2\left(2\hbar\widetilde{\omega}l+mc^2+\hbar\omega\right)}{6\,(k_BT)^2}\right) \end{aligned} \tag{2.136}$$

We neglect the $(k_BT)^{-2}$ term in parentheses and we obtain the high-temperature expansion of the partition function:

$$Z\simeq\frac{(k_BT)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2+\omega^2+2\widetilde{\omega}\omega\tau}}\left(1-\frac{3\,(k_BT)^2\,\lambda}{m^2c^2\left(\widetilde{\omega}^2+\omega^2+2\widetilde{\omega}\omega\tau\right)}\right) \tag{2.137}$$

The first term is the usual partition function for a 2Dfermionic oscillator with a uniform magnetic field in the r-representation. The second term expresses the contribution that comes from the space deformation through the AdS algebra.

At this stage, we can evaluate all the thermodynamic properties of our system (free energy F, mean energy U, specific heat C and entropy S) using their definitions [37]:

$$F = -k_B T \ln Z,\, U = k_B T^2 \frac{\partial \ln Z}{\partial T},\, C = \frac{\partial U}{\partial T}\ and\, S = -\frac{\partial F}{\partial T} \tag{2.138}$$

$$F = -k_B T \ln\left(\frac{(k_B T)^2}{2\hbar m c^2 \sqrt{\widetilde{\omega}^2 + \omega^2 + 2\widetilde{\omega}\omega\tau}}\left(1 - \theta^\tau (k_B T)^2\right)\right) \tag{2.139}$$

$$U = 4k_B T\left[1 - \frac{1}{2\left(1 - \theta^\tau (k_B T)^2\right)}\right] \tag{2.140}$$

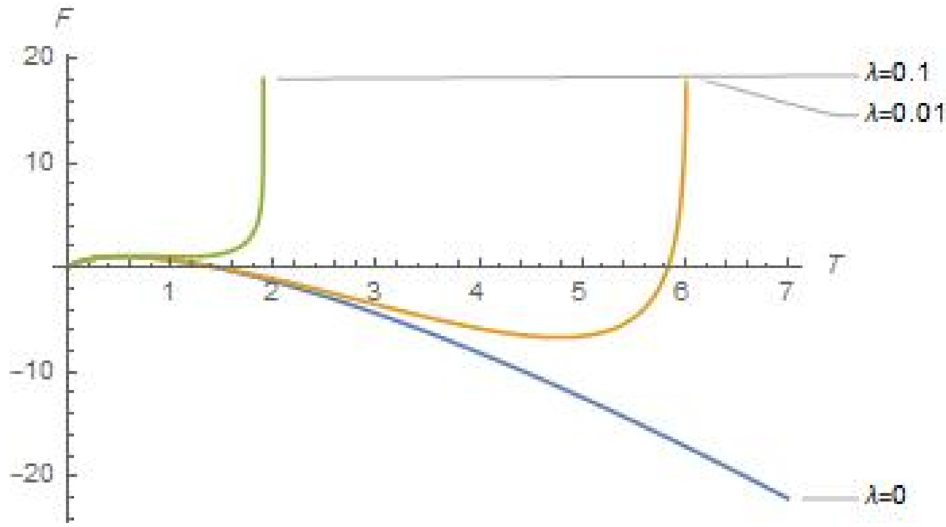
$$C = 4k_B\left[1 - \frac{1 + \theta^\tau (k_B T)^2}{2\left(1 - \theta^\tau (k_B T)^2\right)^2}\right] \tag{2.141}$$

$$S = k_B\left[\frac{2 - 4\theta^\tau (k_B T)^2}{1 - \theta^\tau (k_B T)^2} + \ln\left(\frac{(k_B T)^2}{2\hbar m c^2 \sqrt{\widetilde{\omega}^2 + \omega^2 + 2\widetilde{\omega}\omega\tau}}\left(1 - \theta^\tau (k_B T)^2\right)\right)\right] \tag{2.142}$$

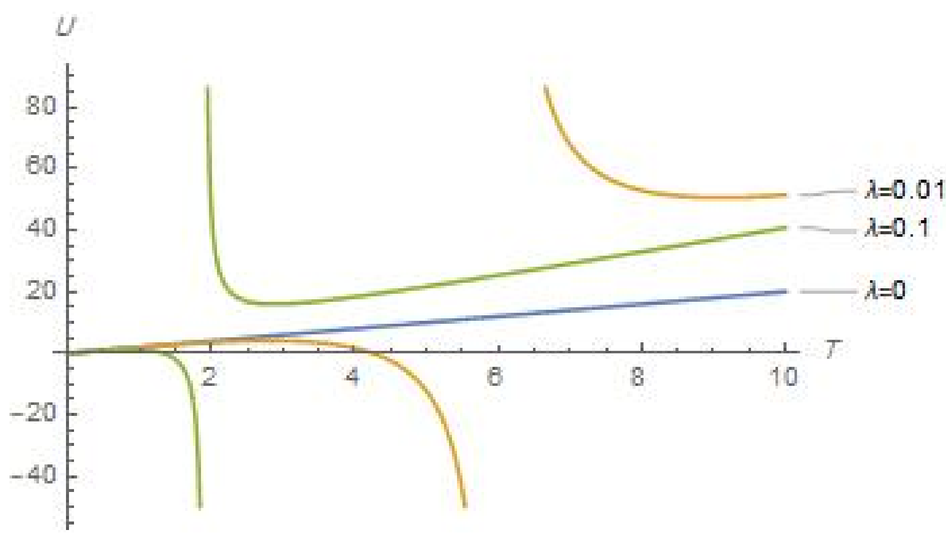
where we have used the parameter $\theta^\tau = \frac{3\lambda}{m^2 c^2\left(\widetilde{\omega}^2 + \omega^2 + 2\widetilde{\omega}\omega\tau\right)}$.

We remark that there is an appearance of a critical point at $T_c = \sqrt{\frac{1+\alpha/4}{3\lambda}} = 1/\sqrt{3\lambda}$ (α is the fine structure constant) given the presence of the factor $1 - \theta^\tau (k_B T)^2$ in the denominator of U, C, and S (in eqs.(2.140), (2.141) and (2.142)) and an additional term regardless the KG spectrum ($2\widetilde{\omega}\omega\tau$).This radically changes the behavior of these thermodynamic properties as we will see in their graphical representations.Furthermore, we notice that our system contains two values of $\tau = \pm 1$,so,we plot the graphs for both values as following

•$\tau = +1$



$Figure.2.5$ The free energy function F according to T for various values of the deformation parameter λ for $\tau = +1$.



$Figure.2.6$ The mean energy U according to T for various values of the deformation parameter λ for $\tau = +1$.

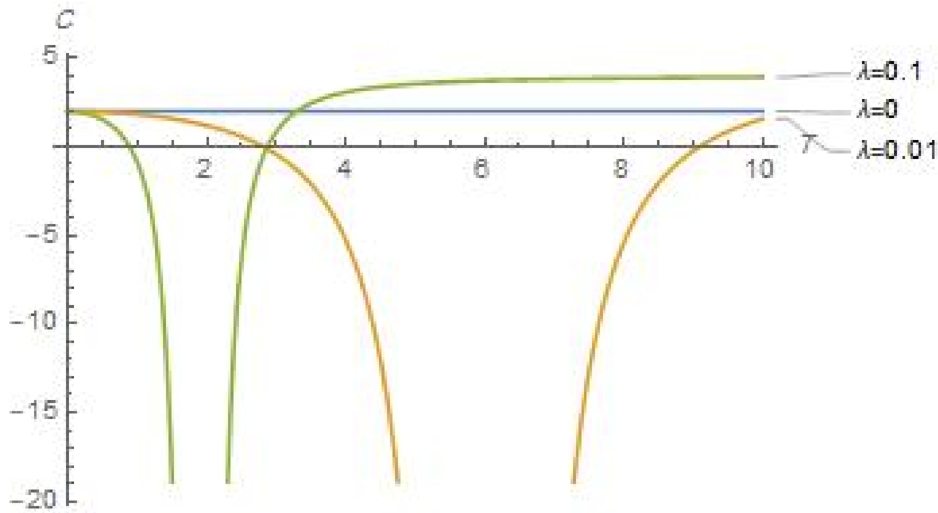


Figure.2.7The specific heat C according to T for various values of the deformation parameter λ for $\tau = +1$.

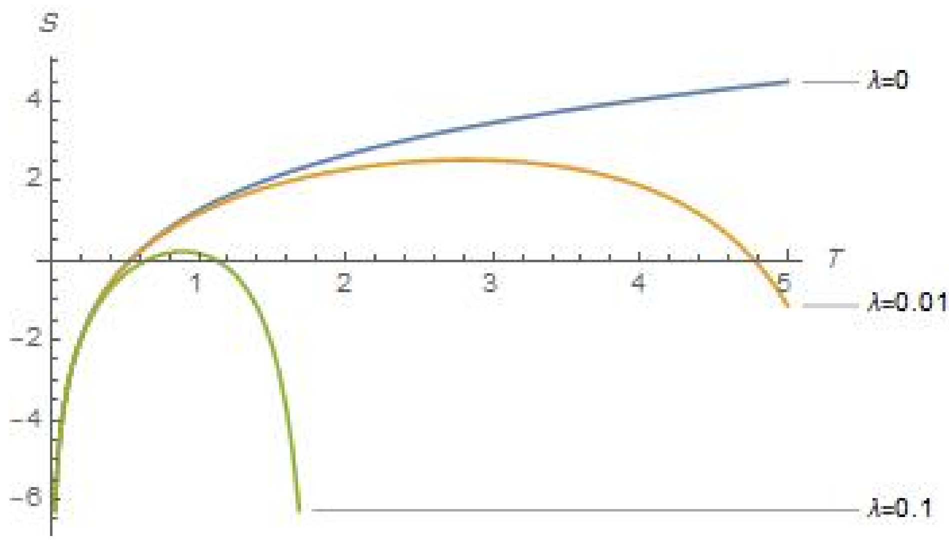
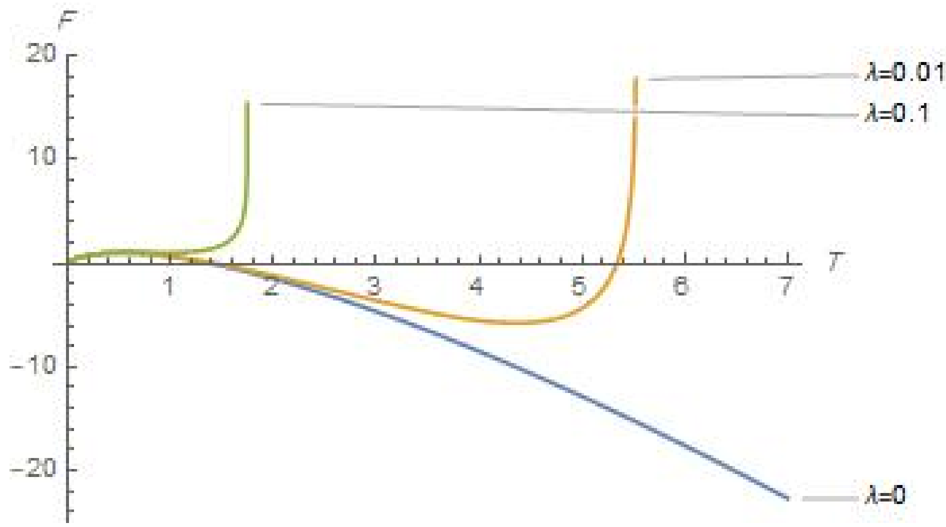
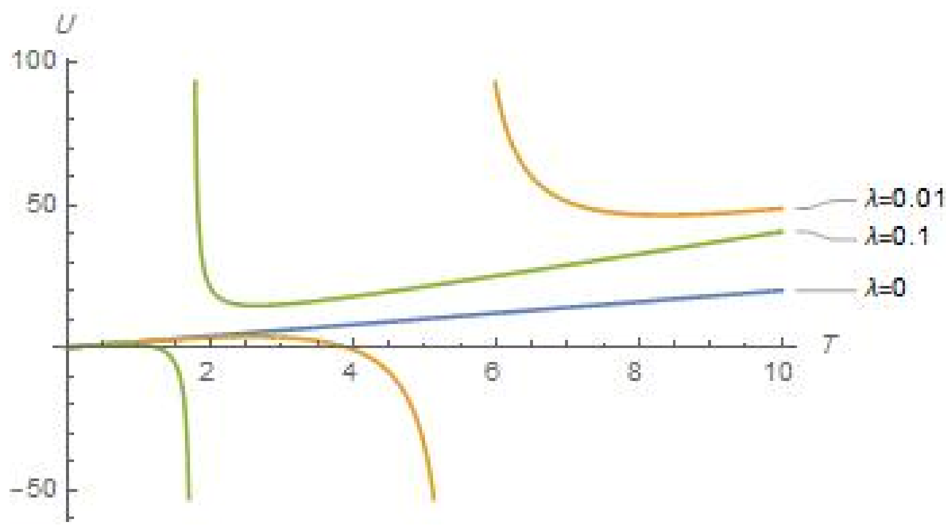


Figure.2.8 The entropy function S according to T for various values of the deformation parameter λ for $\tau = +1$.

$\bullet \tau = -1$



$Figure$.2.9 The free energy function F according to T for various values of the deformation parameter λ for $\tau = -1$.



$Figure$.2.10 The mean energy$U$ according to T for various values of the deformation parameter λ for $\tau = -1$.

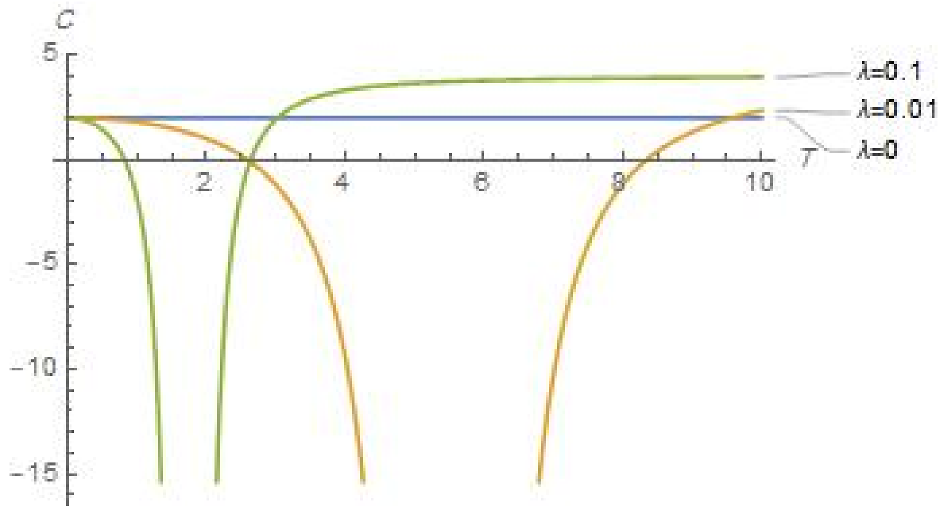


Figure.2.11 The specif heat C according to T for various values of the deformation parameter λ for $\tau = -1$.

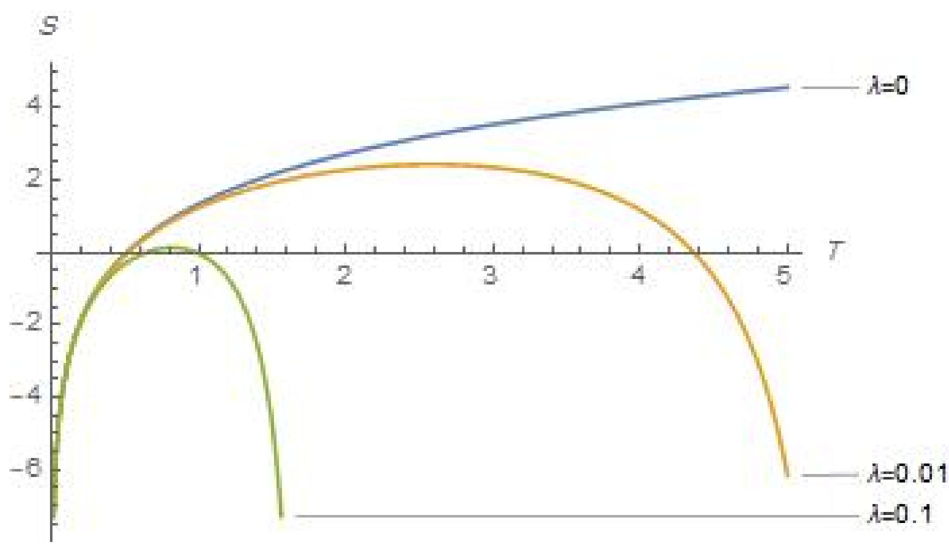


Figure.2.12 The entropy function S according to T for various values of the deformation parameter λ for $\tau = -1$.

Figs.(2.5-2.9) show for both value of τ that the free energy develops fast in the deformed case ($\lambda \neq 0$),however, it decreases to infinity in the normal case; this following a minor increase followed by a reduction common to both situations. The free energy grows with the deformation parameter λ at a certain value of T ,also, we remark that there is a deviation as comparison between the two figures coming from the term $(2\widetilde{\omega}\omega\tau)$ which increase for $(\tau = +1)$ and decrease for $(\tau = -1)$

According to Figs.(2.6-2.10), there is a discontinuity in the mean energy U at T_c, where it declines rapidly approaching T_c and subsequently grows considerably, as in the usual case, and as we mentioned previously there is a deviation too shown in curves.

As shown in Figs.(2.7-2.11), the specific heat C reduces until the discontinuity point T_c, at which point it increases. It is, however, constant for $\lambda = 0$ (i.e., $C = 2k_B$)..

In Figs.(2.8-2.12), we see that the entropy function S grows to a maximum value before decreasing to infinity, in contrast to the usual scenario where it grows continuously with T.

Chapter 3

Relativistic oscillators in Non-Commutative Space

3.0.1 Introduction

It is well known that quantum field theory is fundamentally non-local. The manifestation of this non locality of phenomena becomes very apparent in the high energies. Therefore, any scheme of unification of the interactions of physics should in principle contain effects of non commutativity of space describing the non locality of physical processes. This is one of the arguments recently suggested by string theory in order to unify physics. In addition, this non commutativity of space could in principle absorb the infinities entangling the standard field theories.That is to say, we would be reduced to a Planck renormalization in which instead of quantified radiation-matter exchanges, we would have quantified spatial correlations.

Until the discovery in 1925 of quantum mechanics by Heisenberg, the geometric space of the states of a microscopic system, an atom for example, was enriched by new properties of its coordinates, such as the moment and the position, which no longer switch. Another topology known recently by non-commutative geometry is used.The aim of this non-commutative geometry is to generalize the duality between geometric space and algebra in the more general case where algebra is no longer commutative. This leads to the modification of two fundamental concepts of mathematics, those of space and symmetry, and to the adaptation of the set of appropriate mathematical tools. For example, if we look at the effect of non-commutative geometry on the conventional orbits of the particle, the switching relationship will be implemented by the following modification $[x_i, x_j] = i\theta_{ij}$, which characterises non commutative paces and which expresses the comparison between position observers which generate severe stresses on the value of the non commutative parameter.To this end, much research has been devoted to the study of quantum mechanics deformed by the non-switching geometries of space, for example: extraction of non-commutative quantum mechanics from the non-commutative quantum field theory in the nonrelativistic case [32], the effect of non-commutativity of space on quantum mechanics [69], where the energy spectrum of a general system of non commutative quantum mechanics in a central potential has been corrected by the presence of the non commutativity parameter, the Dirac equation for a spin 1 particle under the action of a constant electromagnetic field that has been studied in the noncommutative space [34],where the production rate of particle pairs was determined by the usual Bogolubov method, and the problem of the KG and Dirac oscillator which was studied and discussed by Mirza in a non commutative space [35], where he showed that the problem for both oscillators behaves similar to the Landau problem in a commutative space.

Based on the usual quantum theory that has been formulated on commutative spaces satisfying the following commutation relationships

$$[\widehat{x}_i, \widehat{x}_j] = 0, [\widehat{p}_i, \widehat{p}_j] = 0, [\widehat{x}_i, \widehat{p}_j] = i\hbar\delta_{ij} \tag{3.1}$$

it is easy to redefine this theory in another non commutative space, by changing the commutation relations in this form

$$[\widehat{x}_i, \widehat{x}_j] = i\theta_{ij} \tag{3.2}$$

$$, [\widehat{p}_i, \widehat{p}_j] = 0, [\widehat{x}_i, \widehat{p}_j] = i\hbar\delta_{ij} \tag{3.3}$$

where $\theta\mu\gamma$ is an antisymmetrical tensor

In order to avoid the problems of unitarity and causality arising in the non commutative theory at such a time $(\theta_{0i} = 0, \theta_{ij} \neq 0)$, we suggest that all time components θ_{0i}should be equal to zero.

Within the same framework of the noncommutative quantum theory, we note that the noncommutative models indicated by the above commutation relationships (3.2) can be achieved in terms of star product, that is to replace directly commutative algebra with the usual product of functions by Moyal algebra with star product

$$(f * g)(x) = \exp\left[\frac{i}{2}\theta_{ab}\partial_{xa}\partial_{yb}\right] f(x)g(y)\mid_{x=y}, \tag{3.4}$$

with $f(x)$, $g(x)$ are two infinitely different arbitrary functions.

On the other hand, it is well known that in the case where $[p_i, p_j] = 0$, the noncommutative quantum mechanics can be reduced to the usual quantum mechanics when the non-commutative coordinate operatorsare expressed in terms of the commutative coordinate operators and their time operators in the following form [35]

$$x_i \rightarrow x_i - \frac{1}{2\hbar}\theta_{ij}p_j \text{ and } p_i \rightarrow p_i, i = 1, 2 \tag{3.5}$$

with the parameters of the antisymmetric tensor θ are chosen as

$$\theta_{ij} = \epsilon_{ijk}\theta_k \text{ and } \theta_3 = \theta \tag{3.6}$$

and other components equal to zero.

On the basis of this fact, the transformation (3.4) can be rewritten in the following condensed form:

$$\mathbf{r} \rightarrow \mathbf{r} + \frac{\boldsymbol{\theta} \times \mathbf{p}}{2\hbar} \text{ with } \theta_{12} = -\theta_{21} = \theta_3 = \theta. \tag{3.7}$$

with $\times$ is the vector product.

The aim of this chapter is to study the relativistic oscillators in a magnetic field in NC space, starting with KG equation ,Dirac equation and at last the DKP equation to solve them in a direct method in order to obtain the energy eigenvalues and the wave function of each equation, in the end of the chapter,we analyse the thermodynamic properties of the two systems.

3.1 Klein Gordon Oscillator in a magnetic field in NC space

The Klein–Gordon oscillator in a two-dimensional ((2+1)-dimensional space-time) commutative space and in a magnetic field has the following form:

$$c^2\left(\mathbf{p}-\frac{e}{c}\mathbf{A}+im\omega\mathbf{r}\right)\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}-im\omega\mathbf{r}\right)\Psi\left(\mathbf{r}\right)=\left(E^2-m^2c^4\right)\Psi\left(\mathbf{r}\right) \tag{3.8}$$

We include the NC algebra definition (3.7) to rewrite this equation in the deformed momentum space:

$$c^2(\mathbf{p}^+\cdot\mathbf{p}^-)\Psi\left(\mathbf{r}\right)=(E^2-m^2c^4)\Psi\left(\mathbf{r}\right) \tag{3.9}$$

with the following definitions:

$$\mathbf{p}^{\pm}=\mathbf{p}^{'}\pm im\omega\left(\mathbf{r}+\frac{\boldsymbol{\theta}\times\mathbf{p}}{2\hbar}\right),\text{with}\mathbf{p}'=\mathbf{p}-\frac{e}{c}\mathbf{B}\times\left(\mathbf{r}+\frac{\boldsymbol{\theta}\times\mathbf{p}}{2\hbar}\right) \tag{3.10}$$

Following a straightforward calculation, we get exactly the following equation:

$$\begin{aligned}&c^2\left[\left(\left(1+\frac{m\widetilde{\omega}\theta}{2\hbar}\right)^2+\frac{m^2\omega^2\theta^2}{4\hbar^2}\right)p^2+\left(m^2\omega^2+\frac{e^2B^2}{4c^2}\right)\mathbf{r}^2\right.\\&\left.-\left(\frac{eB}{c}+\frac{m^2\theta}{\hbar}\left(\omega^2+\widetilde{\omega}^2\right)\right)L_z\right]\Psi\left(\mathbf{r}\right)=\left(E^2-m^2c^4+2m\omega\hbar c^2+m\omega eBc\theta\right)\Psi\left(\mathbf{r}\right)\end{aligned} \tag{3.11}$$

which could be written as

$$\left[p^2+\eta^{\theta}r^2+-\beta^{\theta}L_z-\varepsilon^{\theta}\right]\Psi\left(\mathbf{r}\right)=0 \tag{3.12}$$

with

$$\begin{aligned}\eta^{\theta}&=\frac{1}{\alpha^{\theta}}(m^2\omega^2+\frac{e^2B^2}{4c^2})\text{ and }\varepsilon^{\theta}=\frac{1}{\alpha^{\theta}}\left(\varepsilon+\frac{m\omega eB\theta}{c}\right)\\\alpha^{\theta}&=\left(\left(1+\frac{m\widetilde{\omega}\theta}{2\hbar}\right)^2+\frac{m^2\omega^2\theta^2}{4\hbar^2}\right);\beta^{\theta}=\frac{1}{\alpha^{\theta}}\left(\frac{eB}{c}+\frac{m^2\theta}{\hbar}\left(\omega^2+\widetilde{\omega}^2\right)\right)\end{aligned} \tag{3.13}$$

where ε has the definition as in (2.23)

For the purpose of finding the exact solution of eq(3.12) ,we apply the polar coordinates in position space (r,φ),and we used a seperate form containing the azimuthal quantum number l

$$\Psi\left(\mathbf{r},\varphi\right)=\exp(il\varphi)R(r),\qquad l=0,1,2,..... \tag{3.14}$$

So, the expression of the equation will be such :

$$\left[\left(\frac{d}{dr}\right)^2+\frac{1}{r}\frac{d}{dr}-\frac{l^2}{r^2}-\frac{\eta^\theta r^2}{\hbar^2}+\epsilon^\theta\right]R(r)=0 \tag{3.15}$$

with:

$$\epsilon^\theta{=}\frac{1}{\alpha^\theta}\left(\frac{\varepsilon}{\hbar^2}+\frac{m\omega eB\theta}{c\hbar^2}+\frac{l}{\hbar}\beta^\theta\right) \tag{3.16}$$

In the interest to solve the eq (3.15), we utilize the upcoming transformation;

$$R(r)=\frac{1}{\sqrt{r}}f(r) \tag{3.17}$$

where;

$$\frac{\partial R(r)}{\partial r} = \frac{1}{\sqrt{r}}\left(\frac{\partial}{\partial r}-\frac{1}{2r}\right)f(r) \tag{3.18}$$

$$\frac{\partial^2 R(r)}{\partial r^2} = \frac{1}{\sqrt{r}}\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{3}{4r^2}\right)f(r) \tag{3.19}$$

After replacing the eqs ((3.18) and (3.19)) into eq (3.15), we get:

$$\left[\frac{\partial^2}{\partial r^2}-\frac{l^2-\frac{1}{4}}{r^2}-\frac{r^2}{a^4}+\epsilon\right]f(r)=0 \tag{3.20}$$

taking into consideration that we have putted $a^4=\frac{\hbar^2}{\eta^\theta}$.Then, we propose this variable as

$$z=\left(\frac{r}{a}\right)^2\Rightarrow r^2=za^2 \tag{3.21}$$

so that, we derive it , to procure

$$\frac{\partial f\left(r\right)}{\partial r} = \frac{2}{a^2}r \tag{3.22}$$

$$\frac{\partial^2 f(r)}{\partial r^2} = \frac{2}{a^2}\left(2z\frac{d^2}{dz^2}+\frac{d}{dz}\right)f(r) \tag{3.23}$$

After replacing eqs $(3.22, 3.23)$into eq (3.20) ,which lead us to write it as

$$\left[z\frac{d^2}{dz^2}+\frac{1}{2}\frac{d}{dz}-\frac{l^2-\frac{1}{4}}{4z}-\frac{z}{4}+\frac{a^2}{4}\epsilon^\theta\right]f\left(z\right)=0 \tag{3.24}$$

Finally, in order to get an known form of a differential equation, we use the transformation;

$$f(z)=\exp\left(-\frac{z}{2}\right)z^k w\left(z\right) \tag{3.25}$$

with

$$\frac{df(z)}{dz} = \exp\left(-\frac{z}{2}\right) z^k \left(\frac{k}{z} - \frac{1}{2} + \frac{d}{dz}\right) w(z) \tag{3.26}$$

$$\frac{d^2 f(z)}{dz^2} = \exp\left(-\frac{z}{2}\right) z^k \left[\frac{d^2}{dz^2} + \left(\frac{2k}{z} - 1\right)\frac{d}{dz} + \left(\frac{k^2-k}{z^2} - \frac{k}{z} + \frac{1}{4}\right)\right] w(z) \tag{3.27}$$

after substituting both eqs (3.26 and 3.27) in (3.24), we obtain the following differential equation

$$\left[z\frac{d^2}{dz^2} + \left(2k - z + \frac{1}{2}\right)\frac{d}{dz} + \frac{1}{z}\left(k^2 - \frac{k}{2} - \frac{l^2 - \frac{1}{4}}{4}\right) + \left(\frac{a^2\epsilon}{4} - k - \frac{1}{4}\right)\right] w(z) = 0 \tag{3.28}$$

at this stage, when we put the conditions

$$k^2 - \frac{k}{2} - \frac{l^2 - \frac{1}{4}}{4} = 0 \tag{3.29}$$

we can determine the constant k depending on eq (3.29)

$$k_1 = \frac{1}{2}\left(\frac{1}{2} + l\right) \quad ; \quad k_2 = \frac{1}{2}\left(\frac{1}{2} - l\right) \tag{3.30}$$

we choose the appropriate the value of k which satisfy the condition $f(z) \langle \infty$ to obtain;

$$k_1 = \frac{1}{2}\left(\frac{1}{2} + l\right) \tag{3.31}$$

The solution of this differential equation(3.28) which is regular at the origin $z = 0$ is given in terms of confluent hypergeometric functions as

$$w(z) = C^{'} F(-n, l+1, z) \tag{3.32}$$

According to (3.21) and the transformations,we write

$$f(r) = C_{\theta}^{'} \exp\left(-\frac{r^2}{2a^2}\right)\left(\frac{r}{a}\right)^{2k_1} F\left(-n, l+1, \frac{r^2}{a^2}\right) \tag{3.33}$$

Then, the radial solution of the system become like;

$$R(r) = \frac{C^{'}}{a^{2k_1}} \exp\left(-\frac{r^2}{2a^2}\right) r^{\left(2k_1 - \frac{1}{2}\right)} F\left(-n, l+1, \frac{r^2}{a^2}\right) \tag{3.34}$$

finally, the wave function $\Psi(\mathbf{r},\varphi)$ written as follow

$$\Psi(\mathbf{r},\varphi) = \frac{C^{'}}{a^{2k_1}} \exp\left(-\frac{r^2}{2a^2}\right) r^{\left(2k_1 - \frac{1}{2}\right)} F\left(-n, l+1, \frac{r^2}{a^2}\right) \exp(il\varphi) \tag{3.35}$$

C'_θ is the normalization constant.

Using the condition of the convergence of the solutions at infinity, and from the asymptotic behavior of the confluent series F_1 of eq. (3.35), that for $r \to \infty$ we have ${}_1F_1 \to 0$ which leads to$\Psi \to 0$ at infinity, we obtain the following general quantum condition.

$$n = \frac{a^2 \epsilon^\theta}{4} - k_1 - \frac{1}{4}, n = 0, 1, 2, ... \tag{3.36}$$

Hence, the energy eigenvalues are found as:

$$\begin{aligned} E^\theta_{n,l} &= \pm mc^2 \left[1 - \frac{2\omega\hbar}{mc^2}\Upsilon + \frac{2\hbar}{mc^2}\left\{(2n+l+1)\sqrt{\left(\omega^2 + \widetilde{\omega}^2\right)\Gamma^\theta}\right.\right. \\ &\quad \left.\left. -l\left(\widetilde{\omega} + \frac{m\left(\omega^2 + \widetilde{\omega}^2\right)\theta}{2\hbar}\right)\right\}\right]^{\frac{1}{2}} \end{aligned} \tag{3.37}$$

With

$$\Gamma^\theta = \left(\Upsilon + \frac{m^2\left(\omega^2 + \widetilde{\omega}^2\right)\theta^2}{4\hbar^2}\right) \tag{3.38}$$

The energy eigenvalue of such an oscillator is comparable to the typical Zeeman effect, as was mentioned in [35]

We find that the energy spectrum is exact and not degenerate due to the NC effect.We can see that the obtained energy spectrum has terms related to the parameter of the deformation θ coming from NC space.With full and accurate calculation we conclude that the Klein-Gordon oscillator in uniform magnetic field in non commutative space has the similar behaviors to the Landau problem in commutative space[35][70]

When we put $\theta = 0$,we obtain the KGO energy spectrum in the presence of a magnetic field, moreover, when both $B = \theta = 0$, we get the KGO spectrum in the ordinary space [52].

We can deduce that the magnetic field's influence can balance out the noncommutative space's effect, resulting in a critical magnetic field when the coefficient of L_z is set to zero in eq(3.11), A reliable answer ($\theta \simeq 0 \Rightarrow B \simeq 0$) is provided by

$$B = -\frac{2c\hbar}{e\theta}\left[1 - \sqrt{1 - \frac{m^2\omega^2\theta^2}{\hbar^2}}\right] \tag{3.39}$$

The minus sign $(-)$ indicates that the magnetic momentum is moving against the direction of the vector $\overrightarrow{\theta}$.

By setting $E = mc^2 + E_{nr}$, for the non-relativistic limit and with the assumption $mc^2 \gg E_{nr}$, we can write the non-relativistic spectrum KGO in the NC space like;

$$E^{nr} = \hbar\left(2n+l+1\right)\sqrt{\left(\omega^2+\widetilde{\omega}^2\right)\Gamma^{\theta}} - l\hbar\left(\widetilde{\omega} + \frac{m\left(\omega^2+\widetilde{\omega}^2\right)\theta}{2\hbar}\right) - \hbar\omega\Upsilon \tag{3.40}$$

3.2 Dirac Oscillator in a magnetic field in NC space

Let us consider the stationary Dirac oscillator equation of particle of nonzero mass m

$$\left[c\widehat{\boldsymbol{\alpha}}.(\mathbf{p}+im\omega\widehat{\boldsymbol{\beta}}\mathbf{r}) + \widehat{\boldsymbol{\beta}}mc^2\right]\Psi\left(\mathbf{r}\right) = E\Psi\left(\mathbf{r}\right), \tag{3.41}$$

As the simplifications in chapter two in Dirac equation section's , we write the DO in a magnetic field with the presence of NC space as

$$c^2\left(\left(\mathbf{p}^+.\mathbf{p}^-\right) + i\widehat{\boldsymbol{\sigma}}.(\mathbf{p}^+\times\mathbf{p}^-)\right)\psi_a\left(\mathbf{r}\right) = \left(E^2-m^2c^4\right)\psi_a\left(\mathbf{r}\right) \tag{3.42}$$

We notice that $\mathbf{p}^+$and $\mathbf{p}^-$ are given in eq(3.10)

so that, after a straightforward calculation,we get

$$\left[\left(\Upsilon + \frac{m^2\theta^2}{4\hbar^2}\left(\omega^2+\widetilde{\omega}^2\right) + \frac{m\omega\theta}{\hbar}\left(1+m\widetilde{\omega}\frac{\theta}{2\hbar}\right)\sigma_z\right)p^2 + \left(m^2\left(\widetilde{\omega}^2+\omega^2+2\omega\widetilde{\omega}\sigma_z\right)\right)\mathbf{r}^2 \right.$$
$$\left. -\left(\left(2m\widetilde{\omega} + \frac{m^2\theta}{\hbar}\left(\omega^2+\widetilde{\omega}^2\right)\right) + \left(2m\omega\Upsilon\right)\sigma_z\right)L_z - \left(2m\widetilde{\omega}\hbar + m^2\theta\left(\omega^2+\widetilde{\omega}^2\right)\right)\sigma_z - \varepsilon_\theta^{'}\right]\psi_a\left(\mathbf{r}\right) = 0 \tag{3.43}$$

where

$$\varepsilon_\theta^{'} = \frac{E^2-m^2c^4}{c^2} + 2m\omega\left(\hbar + m\widetilde{\omega}\theta\right);\Upsilon = \left(1+\frac{m\widetilde{\omega}\theta}{\hbar}\right) \tag{3.44}$$

To solve the eq.(3.43), we use the following ansatz $\psi_a\left(\mathbf{r}\right) = e^{il\varphi}R_{nl}(r)\chi_\tau$, and we utilize the polar coordinates of the position and momentum operators, we obtain the following differential equation

$$\left[\left(\frac{d}{dr}\right)^2 + \frac{1}{r}\frac{d}{dr} - \frac{l^2}{r^2} - \frac{\eta_\theta^{`}r^2}{\hbar^2} + \epsilon_\theta^{`}\right]R_{nl}(r) = 0 \tag{3.45}$$

with

$$\eta_\theta^\tau = \frac{m^2\left(\widetilde{\omega}^2 + \omega^2 + 2\omega\widetilde{\omega}\tau\right)}{\alpha_\theta^\tau}$$
$$\epsilon_\theta^\tau = \frac{1}{\alpha_\theta^\tau}\left[\frac{\varepsilon_\theta^{'}}{\hbar^2} + \left(\frac{2m\widetilde{\omega}}{\hbar} + \frac{m^2\theta}{\hbar^2}\left(\omega^2 + \widetilde{\omega}^2\right)\right)\tau + \frac{l}{\hbar}\beta^\theta\right]$$
$$\alpha_\theta^\tau = \left(\Upsilon + \frac{m^2\theta^2}{4\hbar^2}\left(\omega^2 + \widetilde{\omega}^2\right)\right) + \frac{m\omega\theta}{\hbar}\left(1 + m\widetilde{\omega}\frac{\theta}{2\hbar}\right)\tau, \tag{3.46}$$
$$\beta_\theta^\tau = \left(2m\widetilde{\omega} + \frac{m^2\theta}{\hbar}\left(\omega^2 + \widetilde{\omega}^2\right)\right) + \left(2m\omega\Upsilon\right)\tau$$

By the same method as we did in the previous section in this chapter,we obtain the energy eigenvalues of DO in a magnetic field in NC space in the following form:

$$E_{n,l} = \pm mc^2\left[1 - \frac{2\omega\hbar}{mc^2}\left(1 + l\tau\right)\Upsilon + \frac{2\hbar}{mc^2}\left\{\left(2n + l + 1\right)\sqrt{\left(\omega^2 + \widetilde{\omega}^2 + 2\omega\widetilde{\omega}\tau\right)\Gamma_\theta^\tau}\right.\right.$$
$$\left.\left. - \left(l + \tau\right)\left(\widetilde{\omega} + \frac{m\left(\omega^2 + \widetilde{\omega}^2\right)\theta}{2\hbar}\right)\right\}\right]^{\frac{1}{2}} \tag{3.47}$$

With

$$\Gamma_\theta^\tau = \left(\Upsilon + \frac{m^2\left(\omega^2 + \widetilde{\omega}^2\right)\theta^2}{4\hbar^2} - \frac{m\omega\tau}{\hbar}\left(1 + \frac{m\widetilde{\omega}\theta}{2\hbar}\right)\theta\right) \tag{3.48}$$

While n are non-negative integers, we explicitly observe that our eigenvalues are non-degenerated (the spectrum has no degeneracy), this case can be explained by the fact that the particle is restricted to moving in two dimensions, and the third dimension does not contribute in the form of energy. Knowing that, it will be an infinite degeneracy when there is a contribution of an element related to the third dimension, such as k_z or p_z.

We can check our spectrum in many ways such as ; when $\tau = 0$, it could lead us to the KGO spectrum's in NC space as in the expression (3.37). Besides, if we put $\theta = 0$, we obtain the DO in a magnetic field in the ordinary space, more than this, we notice that $(B = \theta = 0)$, it gives us the DO energy spectrum in the commutative space[56].

The non-relativistic limit is obtained, as in the usual case, by setting$E = mc^2 + E_{nr}$ with the assumption that $mc^2 \gg E_{nr}$, the first-order approximation of Taylor expansion is given as

$$E^{nr} = \left(2n + l + 1\right)\hbar\sqrt{\left(\omega^2 + \widetilde{\omega}^2 + 2\omega\widetilde{\omega}\tau\right)\Gamma_\theta^\tau} - \left(l + \tau\right)\hbar\left(\widetilde{\omega} + \frac{m\left(\omega^2 + \widetilde{\omega}^2\right)\theta}{2\hbar}\right)$$
$$-\omega\hbar\left(1 + l\tau\right)\Upsilon \tag{3.49}$$

Now we can write the general form of the wave function Ψ:

$$\Psi\left(r,\varphi\right)=C_{\tau}^{\theta}\left(\frac{r^{2}}{a^{2}}\right)^{\left(\frac{1}{4}-\frac{l}{2}\right)}\exp\left(-\frac{r^{2}}{2a^{2}}\right)L_{n}^{-l}\left(\frac{r^{2}}{a^{2}}\right)\exp\left(il\varphi\right)\chi \tag{3.50}$$

where C_{τ}^{θ} is the normalization constant.

3.3 DKP oscillator in a magnetic field in NC space

The free DKP equation of massive scalar and vector particles m in the NC space has the following form

$$\left[c\beta\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{B}\times\left(\mathbf{r}+\frac{\theta\times\mathbf{p}}{2\hbar}\right)-im\omega\eta^{0}\left(\mathbf{r}+\frac{\theta\times\mathbf{p}}{2\hbar}\right)\right)+mc^{2}\right]\widetilde{\Psi}=E\beta^{0}\widetilde{\Psi} \tag{3.51}$$

where $\Psi\left(\mathbf{r},t\right)=e^{-\frac{iE_{t}}{\hbar}}\widetilde{\Psi}\left(\mathbf{r}\right)$

3.3.1 Scalar particle case

The wave function is a vector with five components for a scalar particle of spin 0, which is written as follows.

$$\widetilde{\Psi}\left(\mathbf{r}\right)=\begin{pmatrix}\mathbf{\Phi}\\ i\psi\end{pmatrix}\text{ with }\mathbf{\Phi}\equiv\begin{pmatrix}\phi\\ \chi\end{pmatrix}\text{ and }\psi\equiv\begin{pmatrix}\psi_{1}\\ \psi_{2}\\ \psi_{3}\end{pmatrix} \tag{3.52}$$

We replace eq.(3.52) with eq.(3.51); we get the coupled system that follows.

$$mc^{2}\phi=E\chi+ic\mathbf{p}^{+}\cdot\psi, \tag{3.53}$$

$$mc^{2}\psi=ic\mathbf{p}^{-}\phi, \tag{3.54}$$

$$mc^{2}\chi=E\phi \tag{3.55}$$

with $\mathbf{p}^{+}$ and $\mathbf{p}^{-}$ defined in eq(3.10)

At this point, the problem can be solved directly by the two-dimensional deformed Klein-Gordon Oscillator in a constant magnetic field with the same differential equation (3.12) when the system is uncoupled in favor of ϕ.

$$\left[p^{2}+\eta^{\theta}r^{2}-\beta^{\theta}L_{z}-\varepsilon^{\theta}\right]\phi=0 \tag{3.56}$$

where the parameters have the similar definitions in eq (3.13)

Taking the identical method as in the KG case, the exact solution of the scalar DKP is taken the following expression

$$E_{n,l}=\pm mc^{2}\left[1-\frac{2\omega\hbar}{mc^{2}}\Upsilon+\frac{2\hbar}{mc^{2}}\left\{(2n+l+1)\sqrt{\left(\omega^{2}+\widetilde{\omega}^{2}\right)\Gamma^{\theta}}\right.\right.$$
$$\left.\left.-l\left(\widetilde{\omega}+\frac{m\left(\omega^{2}+\widetilde{\omega}^{2}\right)\theta}{2\hbar}\right)\right\}\right]^{\frac{1}{2}} \tag{3.57}$$

And to write the general form of the wave function $\Psi(r,\varphi)$ as follows:

$$\Psi(r,\varphi)=C^{\theta}r^{-\frac{1}{2}}e^{-\frac{r^{2}}{2a^{2}}}\left(\frac{r}{a}\right)^{l+\frac{1}{2}}F\left(-n,l+1,\frac{r^{2}}{a^{2}}\right)\exp\left(il\varphi\right) \tag{3.58}$$

Here C^{θ} noted as the normalization constant

3.3.2 vector case

We continue in the same manner as in the preceding one. In this case, the wave function of spin 1 is a vector with ten components noted by $\Psi\left(\mathbf{r}\right)^{T}=\left(i\varphi,\mathbf{A}\left(r\right),\mathbf{B}\left(r\right),\mathbf{C}\left(r\right)\right)$ with A_i, B_i and C_i $(i=1,2,3)$ being ,respectively the components of the vectors $\mathbf{A}\left(r\right),\mathbf{B}\left(r\right),\mathbf{C}\left(r\right)$.The equation (3.51) is reduced to the following system:

$$mc^{2}\varphi=-c\mathbf{p}^{-}\cdot\mathbf{B} \tag{3.59}$$

$$mc^{2}\mathbf{A}=E\mathbf{B}-c\mathbf{p}^{+}\times\mathbf{C} \tag{3.60}$$

$$mc^{2}\mathbf{B}=E\mathbf{A}+c\mathbf{p}^{+}\varphi \tag{3.61}$$

$$mc^{2}\mathbf{C}=-c\mathbf{p}^{-}\times\mathbf{A} \tag{3.62}$$

To decouple the system above , we eliminate φ,$\mathbf{B}$ and $\mathbf{C}$ in terms of $\mathbf{A}$ and we get:

$$\left(E^{2}-m^{2}c^{4}\right)\mathbf{A}=c^{2}\mathbf{p}^{+}\left(\mathbf{p}^{-}\cdot\mathbf{A}\right)-c^{2}\mathbf{p}^{+}\times\left(\mathbf{p}^{-}\times\mathbf{A}\right)-\frac{1}{m^{2}}\mathbf{p}^{+}\left[\mathbf{p}^{-}\cdot\left[\mathbf{p}^{+}\times\left(\mathbf{p}^{-}\times\mathbf{A}\right)\right]\right] \tag{3.63}$$

which can be rewritten in the following form:

$$\left(E^{2}-m^{2}c^{4}\right)\mathbf{A}=c^{2}\left[\left(\mathbf{p}^{+}\cdot\mathbf{p}^{-}\right)\mathbf{A}-\left(\mathbf{p}^{+}\times\mathbf{p}^{-}\right)\times\mathbf{A}\right]-\frac{1}{m^{2}}\mathbf{p}^{+}\left[\mathbf{p}^{-}\cdot\left[\mathbf{p}^{+}\times\left(\mathbf{p}^{-}\times\mathbf{A}\right)\right]\right] \tag{3.64}$$

By a direct calculation, the evaluation of the first two terms of eq.(3.64) gives,

$$\begin{aligned}\left(\mathbf{p}^{+}\cdot\mathbf{p}^{-}\right)\mathbf{A} &= \left[\left(\left(1+\frac{m\widetilde{\omega}\theta}{2\hbar}\right)^{2}+\frac{m^{2}\omega^{2}\theta^{2}}{4\hbar^{2}}\right)p^{2}+\left(m^{2}\left(\omega^{2}+\widetilde{\omega}^{2}\right)\right)r^{2}\right.\\ &\quad\left.-\left(\frac{eB}{c}+\frac{m^{2}\theta}{\hbar}\left(\omega^{2}+\widetilde{\omega}^{2}\right)\right)L_{z}-2m\omega\hbar-\frac{m\omega eB\theta}{c}\right]\mathbf{A} \qquad (3.65)\end{aligned}$$

and

$$\begin{aligned}\left(\mathbf{p}^{+}\times\mathbf{p}^{-}\right)\times\mathbf{A} &= \left[\left(-\frac{2eB}{c}\frac{m\omega}{\hbar}\right)r^{2}+2\left(\frac{eB}{c}+\frac{m^{2}\theta}{\hbar}\left(\omega^{2}+\widetilde{\omega}^{2}\right)+\left(\frac{2m\omega}{\hbar}+\frac{eB\theta m\omega}{c\hbar^{2}}\right)L_{z}\right)\right.\\ &\quad\left.-m\omega\frac{\theta}{\hbar}\left(2+\frac{eB\theta}{c\hbar}\right)p^{2}\right]S_{z}\mathbf{A} \qquad (3.66)\end{aligned}$$

then we inset these results(3.65, 3.66) in eq (3.64),we obtain :

$$\begin{aligned}\varepsilon^{\theta'}\mathbf{A} &= \left[\left(\left(\left(1+\frac{m\widetilde{\omega}\theta}{2\hbar}\right)^{2}+\frac{m^{2}\omega^{2}\theta^{2}}{4\hbar^{2}}\right)+m\omega\frac{\theta}{\hbar}\left(2+\frac{eB\theta}{c\hbar}\right)S_{z}\right)p^{2}\right.\\ &\quad+\left(m^{2}\left(\omega^{2}+\widetilde{\omega}^{2}\right)+2\frac{m\omega eB}{\hbar c}\right)r^{2}\\ &\quad\left.-\left(\left(\frac{eB}{c}+\frac{m^{2}\theta}{\hbar}\left(\omega^{2}+\widetilde{\omega}^{2}\right)\right)+2\left(\frac{2m\omega}{\hbar}+\frac{eB\theta m\omega}{c\hbar^{2}}\right)S_{z}\right)L_{z}\right]\mathbf{A}\\ &\quad-\frac{1}{(mc)^{2}}\mathbf{p}^{+}\left[\mathbf{p}^{-}\cdot\left[\mathbf{p}^{+}\times\left(\mathbf{p}^{-}\times\mathbf{A}\right)\right]\right] \qquad (3.67)\end{aligned}$$

with

$$\varepsilon^{\theta'}=\varepsilon+2\left(\frac{eB}{c}+\frac{m^{2}\theta}{\hbar}\left(\omega^{2}+\widetilde{\omega}^{2}\right)\right)S_{z}+\frac{m\omega eB\theta}{c} \qquad (3.68)$$

The z-components of the orbital angular momentum and the spinor are respectively L_z and S_z.

This last expression (3.67) is equivalent to the behavior of the DKP equation in a commutative space, which describes the movement of a vectorial boson (spin-1) subjected to the action of a constant magnetic field along the z-axis with additional interaction correction that depends on the noncommutativity parameter θ, i.e.: $(S \times P)$ $(\theta \times P)$. As a result, this correction results from the effect of space deformation on the physical system with spin as the Dirac oscillator case in noncommutative space[35], which can be understood as an interaction between the electric dipole moment and magnetic moment.[33][35][71] We see that, unlike the Klein-Gordon case, the initial case of DKPO in noncommutative space of spin 0 (3.56)lacks this correction..

With direct methods, the exact resolution of equation (3.67) remains almost impossible. Therefore, we will restrict ourselves to the analysis of the non-relativistic case, where, since it is of the order m^{-3}, the last term in (3.67) is negligible.

Eq. (3.67)becomes similar to both eqs. (3.12) and (3.56)corresponding to the KG case and to the scalar DKP particle, respectively, we find

$$\begin{aligned} \varepsilon^{\theta'}\mathbf{A} &= \left[\Gamma'^{\theta}p^2 + \left(m^2\left(\omega^2+\widetilde{\omega}^2\right) + 2\frac{eB}{c}\frac{m\omega}{\hbar}\right)r^2 \right. \\ & \left. - \left(\left(\frac{eB}{c} + \frac{m^2\theta}{\hbar}\left(\omega^2+\widetilde{\omega}^2\right)\right) + 2\left(\frac{2m\omega}{\hbar} + \frac{eB\theta m\omega}{c\hbar^2}\right)S_z\right)L_z\right]\mathbf{A} \end{aligned} \tag{3.69}$$

By introducing the eigenvalues of S_z and L_z, we obtain the final expression of the spectrum energetic

$$\begin{aligned} E_{nr} &= \left[(2n+l+1)\,\hbar\sqrt{\left(\omega^2+\widetilde{\omega}^2 \mp 4\widetilde{\omega}\omega\right)\left(\Gamma^{\theta} \pm \frac{2m\omega\theta}{\hbar}\Upsilon\right)} - \hbar l\left(\widetilde{\omega} + \frac{m\left(\omega^2+\widetilde{\omega}^2\right)\theta}{2\hbar} \pm 2\omega\Upsilon\right) \right. \\ & \left. - \left(\hbar\omega + m\omega\widetilde{\omega}\theta \pm \hbar\left(\widetilde{\omega} + \frac{m\theta\left(\omega^2+\widetilde{\omega}^2\right)}{2\hbar}\right)\right)\right] \end{aligned} \tag{3.70}$$

The outcome clearly demonstrates the contributions of each term in equation (3.69), particularly those resulting from the coexistence of spin and deformation. It also includes the additional spin-orbit term $\frac{2eB\theta m\omega}{c\hbar^2}S_zL_z$, which can be interpreted as resulting from the interaction between the two.

3.4 Thermodynamic Properties of KG and DKP oscillators in NC space

In this section, we study the thermodynamic properties of the KG and DKP oscillators in a magnetic field under the influence of NC space. The energy eigenvalues of the KG and DKP oscillators in a magnetic field in NC space are:

$$E_n = \pm mc^2\sqrt{\mu_\theta n + \lambda_\theta}, \quad n = 0, 1, 2, ... \tag{3.71}$$

with

$$\mu_\theta = \frac{4\hbar}{mc^2}\sqrt{\left(\omega^2+\widetilde{\omega}^2\right)\left(\Upsilon+\frac{m^2\left(\omega^2+\widetilde{\omega}^2\right)\theta^2}{4\hbar^2}\right)}; \tag{3.72}$$

$$\lambda_\theta = 1-\frac{2\omega\hbar}{mc^2}\Upsilon+\frac{2\hbar}{mc^2}\left\{(l+1)\sqrt{\left(\omega^2+\widetilde{\omega}^2\right)\left(\Upsilon+\frac{m^2\left(\omega^2+\widetilde{\omega}^2\right)\theta^2}{4\hbar^2}\right)}\right.$$
$$\left. -l\left(\widetilde{\omega}+\frac{m\left(\omega^2+\widetilde{\omega}^2\right)\theta}{2\hbar}\right)\right\} \tag{3.73}$$

Since all thermodynamic quantities can be obtained from the partition function Z, first of all,we calculate the partition function of the system defined under the temperature T as

$$Z(V,T) = \sum_{n=0}^{\infty}\exp\left(-\beta\left(E_n-E_0\right)\right), \tag{3.74}$$

where $\beta = 1/k_BT$, k_B is the Boltzmann constant and E_0 is the ground state energy.For body systems with no internal interactions, the corresponding partition function of the KG should be $(Z)^N$. Here, for simplicity, we focus only on the case $N=1$.

Then, the thermodynamic properties of the physical system, such as free energy, mean energy, specific heat and entropy, can be calculated using the following expressions [37]:

$$F = -\frac{1}{\beta}\ln(Z), \tag{3.75}$$

$$U = -\frac{\partial\ln(Z)}{\partial\beta}, \tag{3.76}$$

$$C = \frac{\partial U}{\partial T} = -k_B\beta^2\frac{\partial U}{\partial\beta}, \tag{3.77}$$

$$S = -\frac{\partial F}{\partial T} = k_B\beta^2\frac{\partial F}{\partial\beta} \tag{3.78}$$

For negative energy states, the partition function Z of the finite temperature KGO T has the form

$$Z = \sum_{n=0}^{\infty}e^{\beta mc^2\left(\sqrt{\mu_\theta n+\lambda_\theta}-\sqrt{\lambda_\theta}\right)} = e^{-\beta mc^2\sqrt{\lambda_\theta}}\sum_{n=0}^{\infty}e^{\beta mc^2\sqrt{\mu_\theta n+\lambda_\theta}} \tag{3.79}$$

We apply the integral test to analyze the convergence of the partition function above. From equation (3.78), it can be seen that the function $f(x) = e^{\beta mc^2\sqrt{\mu_\theta x+\lambda_\theta}}$ is an increasing positive function and the integral is

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} e^{\beta mc^2\sqrt{\mu_\theta x+\lambda_\theta}}dx \tag{3.80}$$

is not convergent.This means that according to the criterion of the integral test, the numerical partition function Z does not converge, and for the positive energy state, the corresponding numerical partition function Z obviously converges.

Based on the above analysis, we can assume that only particles with positive energy can establish a thermodynamic ensemble.

Now we continue with the main object that interests us. Substituting (3.71) in (3.74), we obtain the partition function of the KG in the NC plane as

$$Z = \sum_{n=0}^{\infty} e^{-\beta mc^2\left(\sqrt{\mu_\theta n+\lambda_\theta}-\sqrt{\lambda_\theta}\right)} = e^{\beta mc^2\sqrt{\lambda_\theta}} \sum_{n=0}^{\infty} e^{-\beta mc^2\sqrt{\mu_\theta n+\lambda_\theta}} \tag{3.81}$$

To calculate (3.81), we use Euler-Mclaurin formula

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)\,dx - \sum_{p=1}^{\infty}\frac{1}{(2p)!}B_{2p}f^{(2p-1)}(0), \tag{3.82}$$

with B_{2p} being the Bernoulli numbers and

$$f(x) = e^{-\beta mc^2\sqrt{\mu_\theta x+\lambda_\theta}} \tag{3.83}$$

So we have

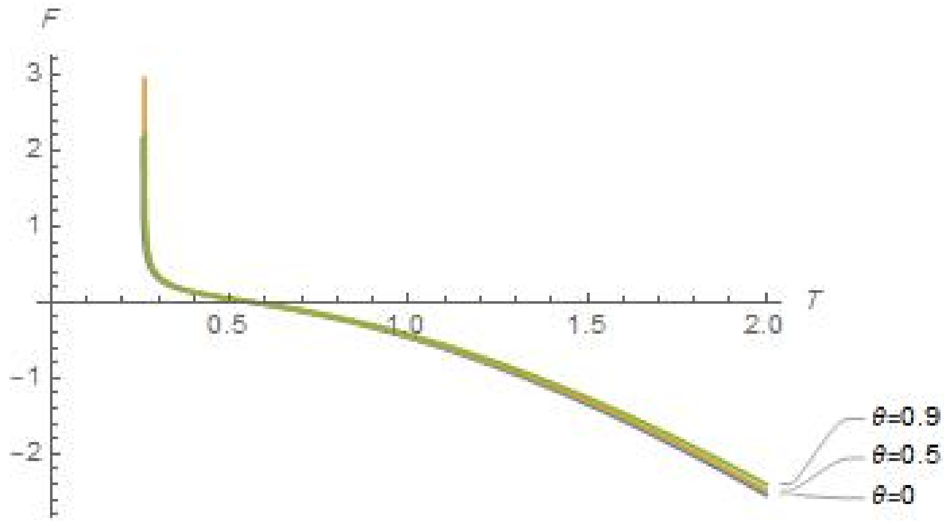
$$\begin{aligned}
\int_0^{\infty} f(x)\,dx = \int_0^{\infty} e^{-\beta mc^2\sqrt{\mu_\theta x+\lambda_\theta}}dx &= \frac{2}{\mu_\theta\beta^2m^2c^4}\left(\beta mc^2\sqrt{\lambda_\theta}+1\right)e^{-\beta mc^2\sqrt{\lambda_\theta}}, \\
f^1(0) &= -\frac{\mu_\theta\beta mc^2}{2\sqrt{\lambda_\theta}}e^{-\beta mc^2\sqrt{\lambda_\theta}}, \\
f^3(0) &= -\frac{\mu_\theta^3\beta mc^2}{8}\left[3\lambda_\theta^{-\frac{5}{2}} + 3\beta mc^2\lambda_\theta^{-2} + \beta^2\left(mc^2\right)^2\lambda_\theta^{-\frac{3}{2}}\right]e^{-\beta mc^2\sqrt{\lambda_\theta}}
\end{aligned} \tag{3.84}$$

Up to B_4, with $B_2=\frac{1}{6}$,$B_4=-\frac{1}{30}$,Combined with (3.84), we obtain the partition function at a finite temperature T of

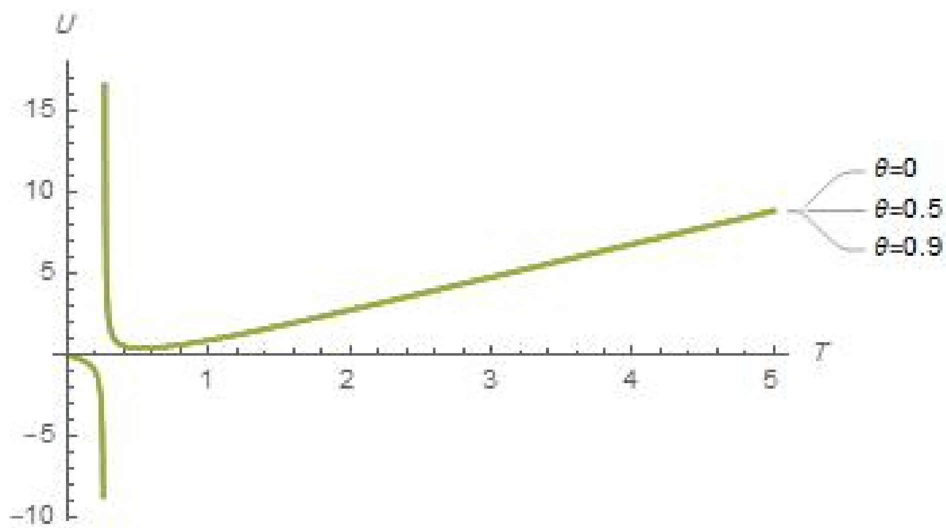
$$Z = \frac{1}{2} + \frac{2\sqrt{\lambda_\theta}}{\mu_\theta}\kappa + \frac{2}{\mu_\theta}\kappa^2 + \left(\frac{\mu_\theta}{24\sqrt{\lambda_\theta}} - \frac{\mu_\theta^3}{1920\sqrt{\lambda_\theta^5}}\right)\frac{1}{\kappa} - \frac{\mu_\theta^3}{1920\lambda_\theta^2}\frac{1}{\kappa^2} - \frac{\mu_\theta^3}{5760\sqrt{\lambda_\theta^3}}\frac{1}{\kappa^3} \tag{3.85}$$

where $\kappa = \frac{1}{\beta mc^2}$,next, we briefly describe the numerical results of the evaluation of thermodynamic functions (that is, free energy, average energy, specific heat, and entropy) through the numerical distribution function Z .

In this section, all profiles of the thermodynamic quantities as a function of dimensionless temperature variable κ for different values of the NC parameter θ , that is, $\theta = 0, 0.5$, and 0.9, are plotted in Figures . In these figures, the Hartree atomic units ($\hbar = c = k_B = m = 1$) are employed and we set $B = \omega = 1$.



$Figure$.3.1 The free energy F function of T for different values of the NC parameter θ.



$Figure$.3.2 The mean energy U function of T for different values of the NC parameter θ.

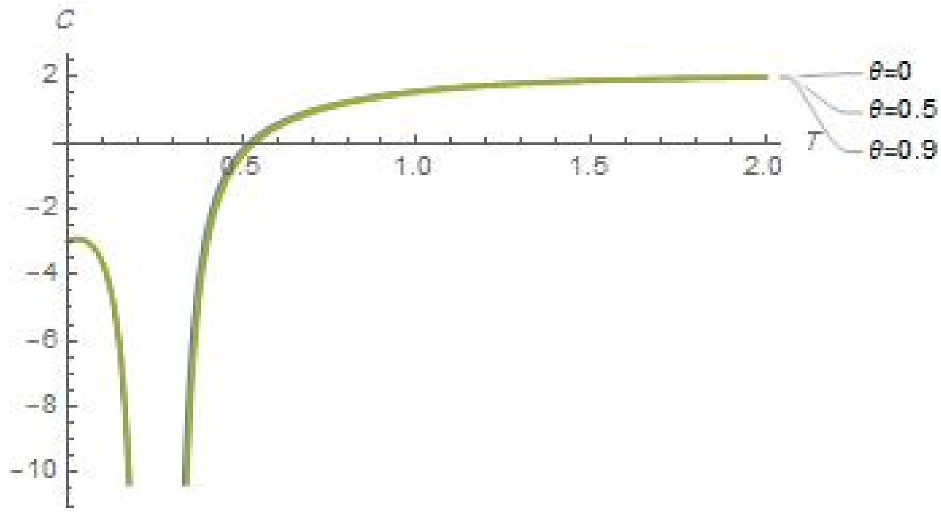


Figure.3.4 The specific heat U function of T for different values of the NC parameter θ.

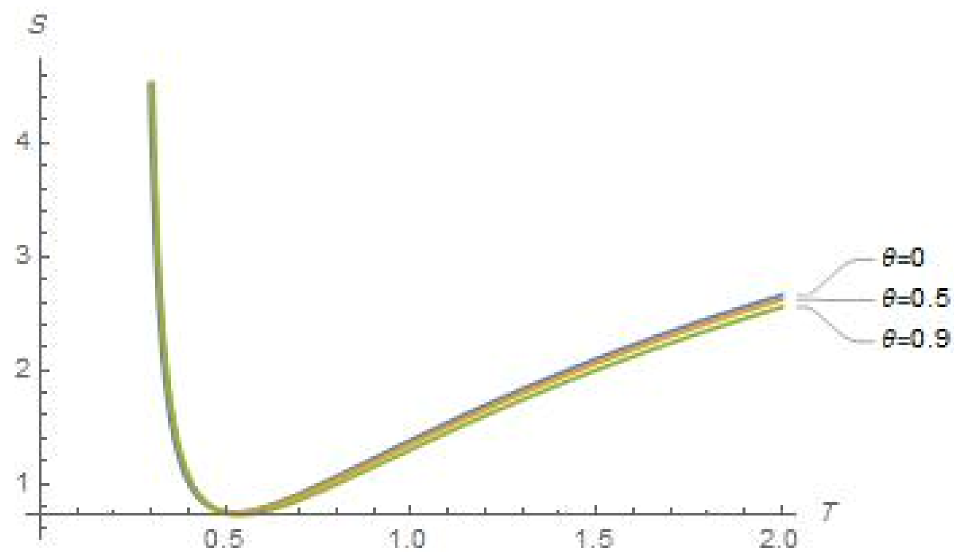


Figure.3.4 The entropy S function of T for different values of the NC parameter θ.

Figure 3.1 depicts the Helmholtz free energy F. For a fixed value of T, we can see that the profile of the curves reduces monotonically with temperature and the free energy increases with increasing NC parameter θ.

In Figure 3.2, we plot the mean energy U versus for various values of the NC parameter

θ, and the results reveal that all of the curves have comparable linear behavior and have extremely similar profiles. Furthermore, we can observe that for a constant value of T, the mean energy diminishes as it expands.

The heat capacity C profile as a function of T for various NC parameters Figure 3.3 depicts a set of values θ. It is discovered that the heat capacity increases initially and then takes a linear behavior as the literature as T increases.($C = 2k_B$)

Figure 3.4 displays the curves of the numerical entropy S versus for different values of the NC parameter θ It shows that the entropy rapidly decreases at first for for a fixed value of T and then slowly grows for large values.of T For a fixed value of T, the entropy decreases when NC parameter θ grows

As a conclusion,we found that the statistical qualities are only slightly affected by NC settings. However, the outcomes can be regarded as a suitable tool to investigate many connected issues.

3.5 Thermodynamic Properties of Dirac oscillators in a magnetic field in NC space

In this section, we analyse the thermodynamic properties of the DO in an external magnetic field in NC space.The energy eigenvalues of the DO in NC space are:

$$E_n = \pm mc^2\sqrt{\mu_{\theta,\tau}n + \lambda_{\theta,\tau}}, \quad n = 0, 1, 2, ... \tag{3.86}$$

with

$$\mu_{\theta,\tau} = \frac{4\hbar}{mc^2}\sqrt{\left(\omega^2 + \widetilde{\omega}^2 + 2\omega\widetilde{\omega}\tau\right)\left(\Upsilon + \frac{m^2\left(\omega^2+\widetilde{\omega}^2\right)\theta^2}{4\hbar^2} - \frac{m\omega\tau}{\hbar}\left(1 + \frac{m\widetilde{\omega}\theta}{2\hbar}\right)\theta\right)}; \tag{3.87}$$

$$\lambda_{\theta,\tau} = 1 + \frac{2\hbar}{mc^2}\left\{(l+1)\sqrt{\left(\omega^2 + \widetilde{\omega}^2 + 2\omega\widetilde{\omega}\tau\right)\left(\Upsilon + \frac{m^2\left(\omega^2+\widetilde{\omega}^2\right)\theta^2}{4\hbar^2} - \frac{m\omega\tau}{\hbar}\left(1 + \frac{m\widetilde{\omega}\theta}{2\hbar}\right)\theta\right)}\right. \tag{3.88}$$

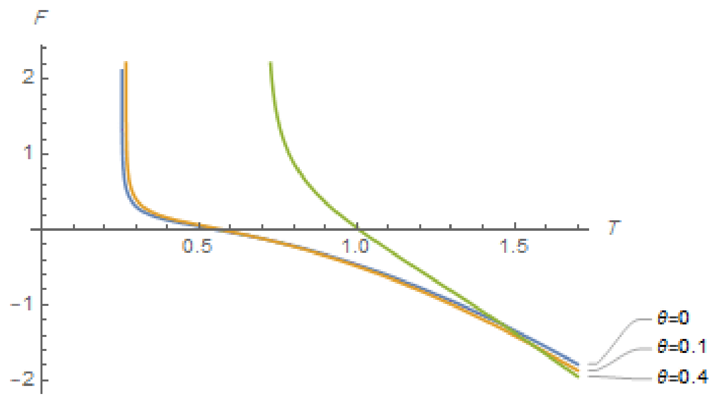
$$\left. - (l+\tau)\left(\widetilde{\omega} + \frac{m\left(\omega^2+\widetilde{\omega}^2\right)\theta}{2\hbar}\right) - \omega\left(1 + l\tau\right)\Upsilon\right\}$$

Alike the KGO case in NC space ,and using the partition function ,

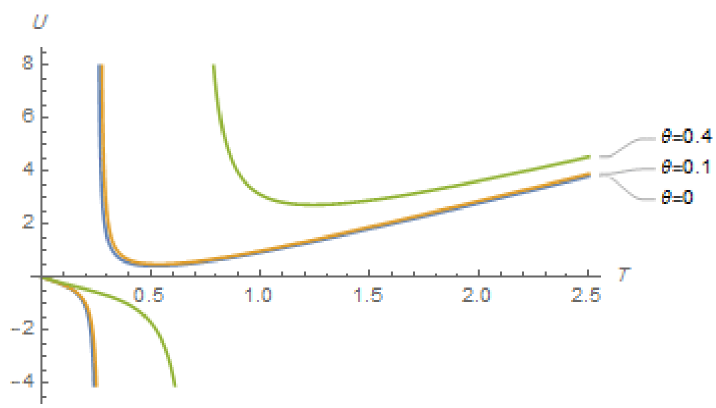
$$Z = \frac{1}{2}+\frac{2\sqrt{\lambda_{\theta,\tau}}}{\mu_{\theta,\tau}}\kappa+\frac{2}{\mu_{\theta,\tau}}\kappa^2+\left(\frac{\mu_{\theta,\tau}}{24\sqrt{\lambda_{\theta,\tau}}}-\frac{\mu_{\theta,\tau}}{1920\sqrt{\lambda_{\theta,\tau}^5}}\right)\frac{1}{\kappa} -\frac{\mu_{\theta,\tau}^3}{1920\lambda_{\theta,\tau}^2}\frac{1}{\kappa^2}-\frac{\mu_{\theta,\tau}^3}{5760\sqrt{\lambda_{\theta,\tau}^3}}\frac{1}{\kappa^3} \quad (3.89)$$

we can plot the thermodynamic features of our system for the free energy, the mean energy, the capacity heat and the entropy as follow

•For $\tau = +1$



$Figure$.3.5 The free energy F function of T for different values of the NC parameter θ for $\tau = +1$.



$Figure$.3.6 The mean energy U function of T for different values of the NC parameter θ for $\tau = +1$.

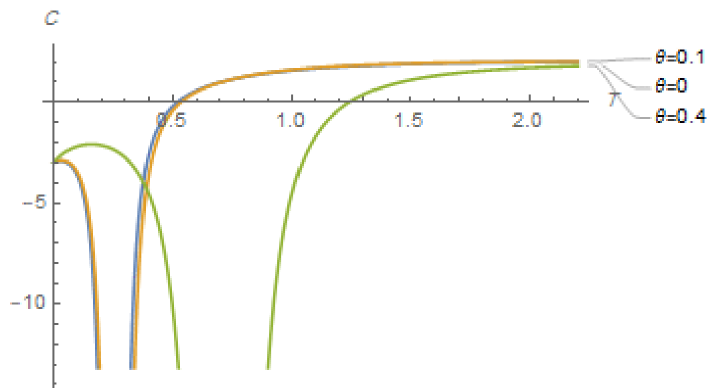


Figure.3.7 The specific heat C function of T for different values of the NC parameter θ for $\tau = +1$.

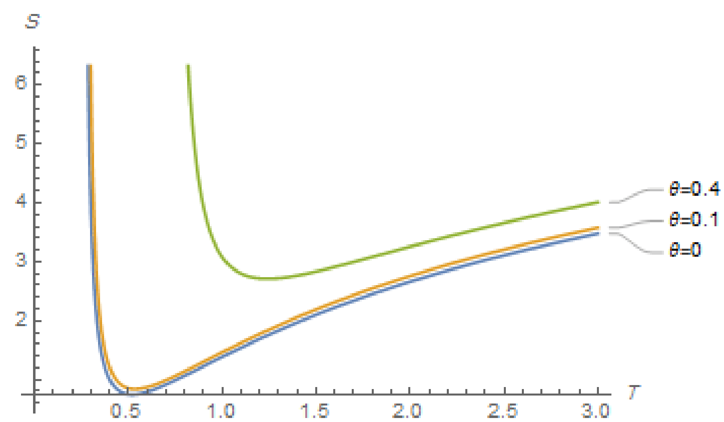
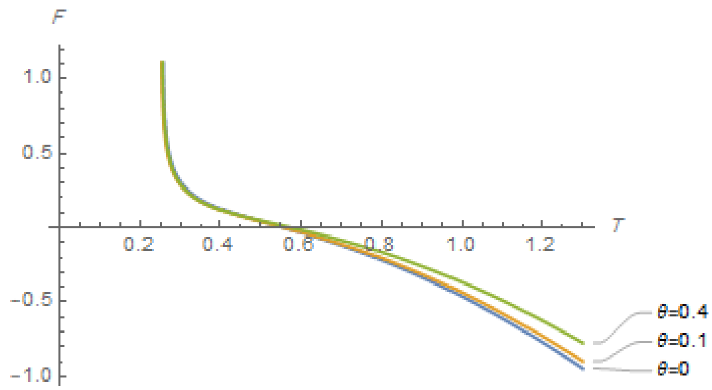
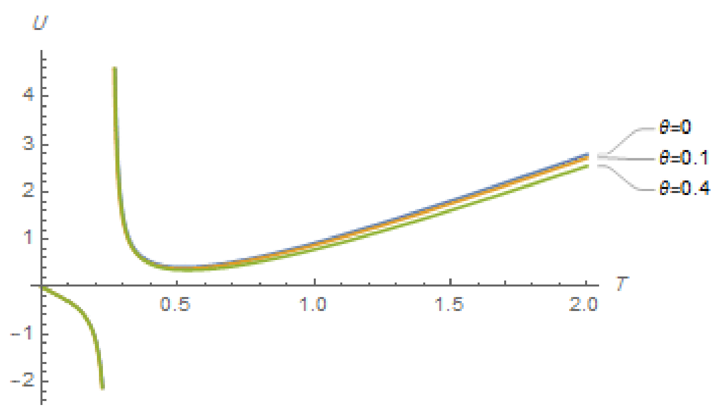


Figure.3.8 The entropy S function of T for different values of the NC parameter θ for $\tau = +1$.

•For $\tau = -1$



$Figure.$3.9 The free energy F function of T for different values of the NC parameter θ for $\tau = -1$.



$Figure.$3.10 The mean energy U function of T for different values of the NC parameter θ for $\tau = -1$.

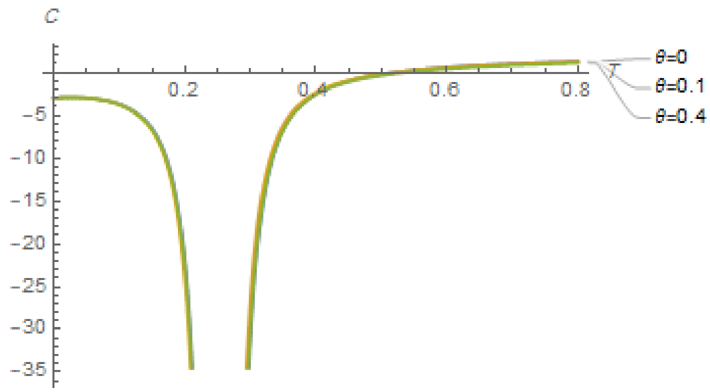


Figure.3.11 The specific heat C function of T for different values of the NC parameter θ for $\tau = -1$.

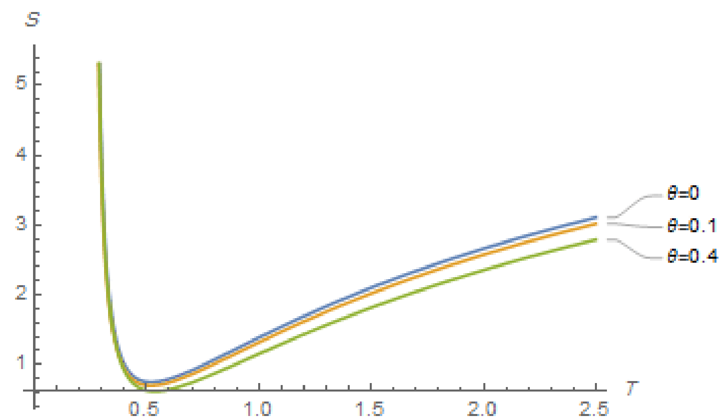


Figure.3.12 The entropy S function of T for different values of the NC parameter θ for $\tau = -1$.

As a discussion to all the figures and taking into consideration that we have plotted the graphs for both values of $\tau = \pm 1$;

Figures (3.5-3.9) depict the Helmholtz free energy F. For a fixed value of T, we can see that the profile of the curves reduces monotonically with temperature and the free energy increases with increasing NC parameter θ, also, for $\tau = +1$, there is an influence shown in curves at increasing of the value of the deformation despite the fact in $\tau = -1$,which is remarked in all the other figures too.

In Figures (3.6-3.10), we plot the mean energy U versus for various values of the NC parameter θ, and the results reveal that all of the curves have comparable linear behavior and have extremely similar profiles. Furthermore, we can observe that for a constant value of T, the mean energy diminishes as it expands.

The heat capacity C profile as a function of T for various NC parameters.Figure (3.7-3.11) depict a set of values θ. It is discovered that the heat capacity increases initially and then takes a linear behavior as the literature as T increases.($C = 2k_B$)

Figures (3.8-3.12) displays the curves of the numerical entropy S versus for different values of the NC parameter θ It shows that the entropy rapidly decreases at first for for a fixed value of T and then slowly grows for large values.of T For a fixed value of T, the entropy decreases when NC parameter θ grows

As the previous comments on the KG thermal properties in NC space, we discovered that NC parameters only have a minor impact on statistical quality on DO in NC space. However, the findings can be used to study a variety of related concerns.

Chapter 4

Relativistic oscillators in a magnetic field in AdS and NC spaces

4.1 Introduction

The purpose of this chapter is to investigate the formulation of a two-dimensional studied relativistic oscillators in the presence of a magnetic field by solving fundamental equations in the framework of relativistic deformed quantum mechanics with EUP and the NC space To do this, we start with KGO in a uniform magnetic field into combination of the AdS and NC spaces,and the same to do with Dirac and DKP equations, and we use the NU method to solve the systems for obtaining the energy eigenvalues and the wave functions.

In the end, we treat the thermodynamic properties of these systems to see how much could be the results influenced by the two spaces.

4.2 KG oscillator in a magnetic field in AdS and NC spaces

We contemplate the stationary relativistic equation with a constant magnetic field in the KG harmonic oscillator bi-dimensional space

$$c^2\left(\mathbf{p}-\frac{e}{c}\mathbf{A}+im\omega\mathbf{r}\right)\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}-im\omega\mathbf{r}\right)\Psi=\left(E^2-m^2c^4\right)\Psi. \tag{4.1}$$

We use the Anti-de Sitter and NC algebra definitions from eqs(2.6,2.7 and 3.7),respectively, we can rewrite this equation in the deformed momentum space

$$c^2(\mathbf{p}^+\cdot\mathbf{p}^-)\Psi(\mathbf{r})=(E^2-m^2c^4)\Psi(\mathbf{r}) \tag{4.2}$$

where;

$$\begin{aligned}\mathbf{p}^{\pm} &= \mathbf{p}^{'}\pm im\omega\left(\frac{\mathbf{r}}{\sqrt{1-\lambda r^2}}+\frac{\boldsymbol{\theta}\times\sqrt{1-\lambda r^2}\mathbf{p}}{2\hbar}\right),\\ \text{with } \mathbf{p}^{'} &= \sqrt{1-\lambda r^2}\mathbf{p}-\frac{e}{c}\mathbf{B}\times\left(\frac{\mathbf{r}}{\sqrt{1-\lambda r^2}}+\frac{\boldsymbol{\theta}\times\sqrt{1-\lambda r^2}\mathbf{p}}{2\hbar}\right)\end{aligned} \tag{4.3}$$

Taking a straightforward computation, we get exactly the following equation:

$$\Bigg[\left(\left(1+\frac{m\widetilde{\omega}\theta}{2\hbar}\right)^2+\frac{m^2\omega^2\theta^2}{4\hbar^2}\right)\left((1-\lambda r^2)p^2+i\hbar\lambda(\mathbf{r}\cdot\mathbf{p})\right)+\left(m^2\left(\omega^2+\widetilde{\omega}^2\right)-m\omega\hbar\lambda\Upsilon\right)\frac{r^2}{1-\lambda r^2}$$
$$-\left(\frac{eB}{c}+m\omega\lambda\theta+\frac{m^2\theta}{\hbar}\left(\omega^2+\widetilde{\omega}^2\right)+\frac{eB\theta^2m\omega}{4c\hbar}\lambda\right)L_z\Bigg]\boldsymbol{\Psi}(\mathbf{r})=\left(\varepsilon+\frac{m\omega eB\theta}{c}\right)\Psi(\mathbf{r}) \tag{4.4}$$

with

$$\varepsilon=\frac{(E^2-m^2c^4)}{c^2}+2m\omega\hbar \tag{4.5}$$

Eq(4.4) could be take the form

$$\left[(1-\lambda r^2)p^2+\eta_\theta^\lambda\frac{r^2}{1-\lambda r^2}+i\hbar\lambda(\mathbf{r}\cdot\mathbf{p})-\beta_\theta^\lambda L_z-\varepsilon_\theta^\lambda\right]\Psi(\mathbf{r})=0 \tag{4.6}$$

with

$$\eta_\theta^\lambda = \frac{1}{\alpha_\theta}\left(m^2\left(\omega^2+\widetilde{\omega}^2\right)-m\omega\hbar\lambda\Upsilon\right) \tag{4.7}$$

$$\varepsilon_\theta^\lambda = \frac{1}{\alpha_\theta}\left(\varepsilon+\frac{m\omega eB\theta}{c}\right) \tag{4.8}$$

$$\alpha_\theta = \left(1+\frac{m\widetilde{\omega}\theta}{2\hbar}\right)^2+\frac{m^2\omega^2\theta^2}{4\hbar^2}; \tag{4.9}$$

$$\beta_\theta^\lambda = \frac{1}{\alpha_\theta}\left(\frac{eB}{c}+m\omega\lambda\theta+\frac{m^2\theta}{\hbar}\left(\omega^2+\widetilde{\omega}^2\right)+\frac{eB\theta^2m\omega}{4c\hbar}\lambda\right) \tag{4.10}$$

We can test the shape of this differential equation (4.6)as follows:

•$\theta=0$ (without the deformation of NC space), we obtain the KG equation in AdS space shown in eq(2.22)

$\bullet\lambda = 0$ (without the deformation of AdS space), we obtain the KG equation in NC space shown in eq(3.12)

To get the exact solution of eq(4.6), we use the polar coordinates in position space (r,φ),and we used a seperate form containing the azimuthal quantum number l

$$\Psi\left(r,\varphi\right) = \exp(il\varphi)R(r), \qquad l = 0, 1, 2, ... \tag{4.11}$$

So, the expression of the equation (4.6)will be such

$$\left[\left(\sqrt{1-\lambda r^2}\frac{d}{dr}\right)^2 + \frac{1-\lambda r^2}{r}\frac{d}{dr} - \frac{l^2\left(1-\lambda r^2\right)}{r^2} - \frac{\eta_\theta^\lambda r^2}{\hbar^2\left(1-\lambda r^2\right)} + \epsilon_\theta^\lambda\right]R(r) = 0 \tag{4.12}$$

with:

$$\epsilon_\theta^\lambda = \frac{1}{\alpha_\theta}\left(\frac{\varepsilon}{\hbar^2} + \frac{m\omega eB\theta}{c\hbar^2} + \frac{l}{\hbar}\beta_\theta^\lambda\right) \tag{4.13}$$

In the interest for solving eq.(4.12), we use the transformation as it mentioned in eq (2.26).To solve the differential equation, we use the same steps as the transformations and the Nikiforov Uvarov approach that were covered in chapter 2.and we consider the conditions

$$\Lambda_n = k_2 - \frac{l}{2} = n\left(n + \mu_{\lambda,\theta} + l + \frac{1}{2}\right), n = 0, 1, 2, ... \tag{4.14}$$

$$\text{and } \mu_{\lambda,\theta} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\eta_\theta^\lambda}{\hbar^2\lambda^2}} \tag{4.15}$$

To procure the energy spectrum of KG oscillator in both deformations as

$$E_{n,l}\left(\lambda,\theta\right) = \pm mc^2\left[1 - \frac{2\omega\hbar}{mc^2}\Upsilon + \frac{2\hbar}{mc^2}\left\{(2n+l+1)\sqrt{\Phi\Gamma^\theta} + \frac{\lambda\hbar}{2m}\left(4n\left(n+l+1\right)\Gamma^\theta + 2l + 1\right) - l\left(\widetilde{\omega} + \frac{m\left(\omega^2 + \widetilde{\omega}^2\right)\theta}{2\hbar} + \frac{\omega}{2}\Upsilon\lambda\theta\right)\right\}\right]^{\frac{1}{2}} \tag{4.16}$$

With

$$\Gamma^\theta = \left(\Upsilon + \frac{m^2\left(\omega^2 + \widetilde{\omega}^2\right)\theta^2}{4\hbar^2}\right); \Upsilon = \left(1 + \frac{m\widetilde{\omega}\theta}{\hbar}\right) \tag{4.17}$$

$$\Phi = \left(\omega - \frac{\lambda\hbar}{2m}\right)^2 + \widetilde{\omega}^2 - \widetilde{\omega}\left(\omega - \frac{\lambda\hbar}{m}\right)\lambda\theta + \frac{\left(\omega^2 + \widetilde{\omega}^2\right)\lambda^2\theta^2}{16} \tag{4.18}$$

The energy spectrum depends, as it should, on the deformations parameters θ and λ which add contributions coming from the interactions of the noncommutativity with the angular momentum and the oscillator with the AdS deformation's parameter λ. These additional contributions are similar to those coming from the interaction of the system with the magnetic field

We can see that the presence of both parameters λ and θ with a magnetic field breaks the degeneracy of the spectrum of energy. In addition, the presence of a term on n^2 shows the existence of hard confinement.

We can analyze the shape of the energy spectrum as follows:

•When $\theta = 0$, It corresponds to same spectrum of the deformed 2D KG oscillator under a uniform magnetic field with Snyder-de Sitter algebra if we ignore the effect of Snyder algebra ($\alpha_2 = 0$ and $\alpha_1 = \lambda$)[30] and, if we study the limit $\lambda \to 0$, we obtain the exact result of the 2D KG oscillator under a magnetic field without deformation. We also note that the result is strictly consistent with the usual KG oscillator when both the deformed parameter and the magnetic field are absent (i.e. $\lambda = B = 0$) [52], besides, when the deformation's parameter of NC space goes to zero, we obtain the energy spectrum in our contribution[72].

•When $\lambda = 0$,we get the exact energy spectrum of the 2D KGO in a magnetic field in NC space.Moreover, if we put ($\theta = B = 0$),it gives us the 2D KGO spectrum in the ordinary case.

we compare our result with the previous ones (chapter 2 and 3) and discuss the corresponding special cases. It shows that the energy eigenvalues increase monotonically with the magnetic field variable for the AdS parameter and the NC space parameter, respectively, and for one principal quantum number, the energy increases with the increase of the azimuthal quantum number.

To deduce the complete expression of the wave function $\Psi(r,\varphi)$,we take the similar procedure as in KGO in a magnetic field in AdS space,and in terms of the variables r and φ, we can write the general form of the wave function Ψ:

$$\Psi(r,\varphi) = C_n^{\lambda,\theta} 2^{\frac{l}{2}} e^{il\varphi} \left(1 - \lambda r^2\right)^{\frac{\mu_{\lambda,\theta}}{2}} \left(\lambda r^2\right)^{\frac{l}{2}} P_n^{(l,\mu_{\lambda,\theta}-1/2)} \left(1 - 2\lambda r^2\right) \tag{4.19}$$

$C_n^{\lambda,\theta}$ the normalization constant.

The nonrelativistic spectrum is obtained using the usual approximation, to deduce

$$\begin{aligned} E_{n,l}^{nr}(\lambda,\theta) &= \left[\hbar\left(2n+l+1\right)\sqrt{\Phi\Gamma^{\theta}}+\frac{\lambda\hbar^{2}}{2m}\left(4n\left(n+l+1\right)\Gamma^{\theta}+2l+1\right)\right. \\ &\left.-l\hbar\left(\widetilde{\omega}+\frac{m\left(\omega^{2}+\widetilde{\omega}^{2}\right)\theta}{2\hbar}+\frac{\omega}{2}\Upsilon\lambda\theta\right)-\omega\hbar\Upsilon\right] \end{aligned} \tag{4.20}$$

The effect of the noncommutative space is also able to counteract the effect of the magnetic field in the case of anomalous Zeeman effect when L_z is equal to zero.

Here we obtain another critical magnetic field value as (we put $L_z = 0$ in eq (4.4))

$$B_c = -\frac{2c\hbar}{e\theta}\left[1-\sqrt{1-\frac{m^2\omega^2\theta^2}{\hbar^2}-\frac{m\omega\lambda\theta^2}{2\hbar}\left(1+\frac{m\omega\lambda\theta^2}{8\hbar}\right)}\right]-\frac{m\omega c\lambda\theta}{2e} \tag{4.21}$$

We can remark that the first two terms into the root are similar to the critical magnetic field value in NC space as it mentioned in eq(3.39), and the additional terms are coming from the combination of both spaces AdS and NC.

For small values of θ,we get

$$B = -\frac{m^2\omega^2 c\theta}{e\hbar}-\frac{m\omega c\lambda\theta}{e} \tag{4.22}$$

we can notice that the first term is related to the NC space at it mentioned in [35], and the combination of two deformed spaces comes in the additional term.And the minus sign indicates that $e\rangle 0,\overrightarrow{B}$ and $\overrightarrow{\theta}$ have inverse directions.

4.3 Dirac Oscillators in a magnetic field in AdS and NC spaces:

We consider the stationary Dirac oscillator equation of particle of nonzero mass m

$$\left[c\widehat{\boldsymbol{\alpha}}.(\mathbf{p}+im\omega\widehat{\boldsymbol{\beta}}\mathbf{r})+\widehat{\boldsymbol{\beta}}mc^{2}\right]\Psi=E\Psi, \tag{4.23}$$

By taking account of the definition of the anti-de Sitter and the noncommutative algebras from eqs(2.6,2.7 and 3.7),respectively, the two dimensional deformed stationary Dirac oscillator equation can be put then,

$$c\widehat{\boldsymbol{\sigma}}.\mathbf{p}^{+}\psi_b = \left(E - mc^2\right)\psi_a \tag{4.24}$$

$$c\widehat{\boldsymbol{\sigma}}.\mathbf{p}^{-}\psi_a = \left(E + mc^2\right)\psi_b \tag{4.25}$$

After simplifications, we write

$$c^2\left(\left(\mathbf{p}^{+}\cdot\mathbf{p}^{-}\right) + i\widehat{\boldsymbol{\sigma}}.(\mathbf{p}^{+}\times\mathbf{p}^{-})\right)\psi_a = \left(E^2 - m^2c^4\right)\psi_a \tag{4.26}$$

here $\mathbf{p}^{+}$ and $\mathbf{p}^{-}$ have the definitions in eq(4.3) .

By calculation ,we procure:

$$\left[\left((1-\lambda r^2)p^2 + i\hbar\lambda(\mathbf{r}\cdot\mathbf{p})\right) + \eta^{\tau}_{\lambda,\theta}\frac{r^2}{1-\lambda r^2} - \beta^{\tau}_{\lambda,\theta}L_z - \varepsilon^{\tau}_{\lambda,\theta}\right]\psi_a(\mathbf{r}) \tag{4.27}$$

With the abrreviations;

$$\begin{aligned}
\eta^{\tau}_{\lambda,\theta} &= \frac{1}{\alpha^{\tau}_{\theta}}\left[m^2\left(\omega^2+\widetilde{\omega}^2\right) - \left(\frac{eB\theta m\omega}{2c} + m\omega\hbar\right)\lambda\right.\\
&\quad \left. - \left(\frac{eB}{2c}\left(\lambda\hbar - 2m\omega + \frac{eB}{4c}\theta\lambda\right) + \frac{m^2\omega^2\theta}{2}\lambda\right)\sigma_z\right] \qquad (4.28)\\
\beta^{\tau}_{\lambda,\theta} &= \frac{1}{\alpha^{\tau}_{\theta}}\left(\frac{eB}{c} + m\omega\lambda\theta + \frac{m^2\theta}{\hbar}\left(\omega^2+\widetilde{\omega}^2\right) + \frac{eB\theta^2 m\omega}{4c\hbar}\lambda\right.\\
&\quad \left. + \left(\lambda\hbar + \frac{eB}{2c}\theta\lambda + 2m\omega + \frac{eB\theta m\omega}{c\hbar} + \frac{m^2\theta^2\lambda}{4\hbar}\left(\omega^2+\widetilde{\omega}^2\right)\right)\sigma_z\right) \qquad (4.29)\\
\varepsilon^{\tau}_{\lambda,\theta} &= \frac{1}{\alpha^{\tau}_{\theta}}\left(\varepsilon + \frac{m\omega eB\theta}{c} + \left(\frac{eB\hbar}{c} + m^2\theta\left(\omega^2+\widetilde{\omega}^2\right)\right)\sigma_z\right) \qquad (4.30)\\
\alpha^{\tau}_{\theta} &= \left(\Gamma^{\theta} + \frac{m\omega}{\hbar}\left(1 + \frac{m\widetilde{\omega}\theta}{2\hbar}\right)\sigma_z\right) \qquad (4.31)
\end{aligned}$$

To solve the eq.(4.27), we use the ansatz $\psi_a(\mathbf{r}) = e^{il\varphi}R_{nl}(r)\chi_{\tau}$, where n is the radial quantum number, l and $\tau = \pm 1$ are, respectively, the eigenvalues of angular momentum and spin operators, and $\chi^{T}_{+1} = (1,0)$, $\chi^{T}_{-1} = (0,1)$ are the spin functions.

Using the polar coordinates of the position and momentum operators, we obtain the following differential equation for the radial part of the wave function:

$$\left[\left(\sqrt{1-\lambda r^2}\frac{d}{dr}\right)^2 - \frac{l^2\left(1-\lambda r^2\right)}{r^2} - \frac{\eta^{\tau}_{\lambda,\theta}r^2}{\hbar^2\left(1-\lambda r^2\right)} + \epsilon^{\tau}_{\lambda,\theta}\right]R_{nl}(r) = 0 \tag{4.32}$$

with

$$\epsilon^{\lambda}_{\theta} = \frac{1}{\alpha_{\theta}}\left(\frac{\varepsilon}{\hbar^2} + \frac{m\omega eB\theta}{c\hbar^2} + \left(\frac{eB}{c\hbar} + \frac{m^2\theta}{\hbar^2}\left(\omega^2+\widetilde{\omega}^2\right)\right)\tau + \frac{l}{\hbar}\beta^{\lambda}_{\theta}\right) \tag{4.33}$$

As in chapter 2,we apply the steps in using the transformations and the NU method for finding the energy spectrum as:

$$E_{n,l}^{\tau}(\lambda,\theta)=\pm mc^{2}\left[1+\frac{2\hbar}{mc^{2}}\left\{(2n+l+1)\sqrt{\left(\Phi+\left(2\widetilde{\omega}\left(\omega-\frac{\lambda\hbar}{2m}-\frac{\widetilde{\omega}\theta\lambda}{4}\right)-\frac{\omega^{2}\theta}{2}\lambda\right)\tau\right)\Omega^{\tau}+}\right.\right.$$

$$\frac{\lambda\hbar}{2m}\left(4n\left(n+l+1\right)\Omega^{\tau}+\left(2-\tau\right)l+1\right)-\omega\Upsilon\left(l\tau+1\right) \tag{4.34}$$

$$\left.\left.-\left(l+\tau\right)\left(\widetilde{\omega}+\frac{m\left(\omega^{2}+\widetilde{\omega}^{2}\right)\theta}{2\hbar}+\frac{\omega}{2}\Upsilon\lambda\theta\right)\right\}\right]^{\frac{1}{2}} \tag{4.35}$$

with

$$\Omega^{\tau}=\left(\Gamma^{\theta}+\frac{m\omega\tau}{\hbar}\left(1+\frac{m\widetilde{\omega}\theta}{2\hbar}\right)\theta\right) \tag{4.36}$$

$$\Phi=\left(\omega-\frac{\lambda\hbar}{2m}\right)^{2}+\widetilde{\omega}^{2}-\widetilde{\omega}\left(\omega-\frac{\lambda\hbar}{m}\right)\lambda\theta+\frac{\left(\omega^{2}+\widetilde{\omega}^{2}\right)\lambda^{2}\theta^{2}}{16} \tag{4.37}$$

Where Γ^{θ} defined in eq (4.17)

The energy spectrum depends, as it should, on the deformations parameters θ and λ which add contributions coming from the interactions of the noncommutativity with the angular momentum and the oscillator with the AdS deformation's parameter λ. These additional contributions are similar to those coming from the interaction of the system with the magnetic field

We can see that the presence of both parameters λ and θ with a magnetic field breaks the degeneracy of the spectrum of energy. In addition, the presence of a term on n^2 shows the existence of hard confinement.

We can analyze the shape of the energy spectrum as follows:

•When $\theta=0$, It corresponds to same spectrum of the deformed 2D DO under a uniform magnetic field with Anti-de Sitter algebra as in eq(2.77)

•When $\lambda=0$,we get the exact energy spectrum of the 2D DO in a magnetic field in NC space illustrated in eq(3.47).Moreover, if we put $(\theta=B=0)$,it gives us the 2D DO spectrum in the ordinary case

In terms of variables r and φ, we deduce the general form of the wave function;

$$\Psi\left(\mathbf{r},\varphi\right)=C_{\lambda,\theta}^{\backprime}2^{\frac{l}{2}}e^{il\varphi}\left(1-\lambda r^{2}\right)^{\frac{\mu_{\lambda,\theta}^{\tau}}{2}}\left(\lambda r^{2}\right)^{\frac{l}{2}}P_{n}^{\left(l,\mu_{\lambda,\theta}^{\tau}-\frac{1}{2}\right)}\left(1-2\lambda r^{2}\right)\chi_{\tau} \tag{4.38}$$

where $\mu^{\tau}_{\lambda,\theta} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\eta^{\tau}_{\lambda,\theta}}{\hbar^2\lambda^2}}$ and $C^{\backprime}_{\lambda,\theta}$ presents the normalization constant.

For the nonrelativistic limit, we set $E = mc^2 + E_{nr}$ with the assumption that $mc^2 \gg E_{nr}$, so,we write the nonrelativistic spectrum of the deformed DO in AdS and NC spaces as

$$\begin{aligned} E^{\tau}_{nr}(\lambda,\theta) &= \Bigg[\hbar(2n+l+1)\sqrt{\left(\Phi + \left(2\widetilde{\omega}\left(\omega - \frac{\lambda\hbar}{2m} - \frac{\widetilde{\omega}\theta\lambda}{4}\right) - \frac{\omega^2\theta}{2}\lambda\right)\tau\right)\Omega^{\tau}} \\ &+ \frac{\lambda\hbar^2}{2m}\left(4n(n+l+1)\Omega^{\tau} + (2-\tau)l + 1\right) \\ &- (l+\tau)\hbar\left(\widetilde{\omega} + \frac{m\left(\omega^2 + \widetilde{\omega}^2\right)\theta}{2\hbar} + \frac{\omega}{2}\Upsilon\lambda\theta\right) - \omega\hbar\Upsilon(l\tau+1)\Bigg] \end{aligned} \tag{4.39}$$

4.4 DKP oscillators in a magnetic field in AdS and NC spaces:

The DKP equation of massive scalar and vector particles m in the AdS and NC spaces has the following form

$$\begin{aligned} &\Bigg[\Bigg(c\beta\cdot\sqrt{1-\lambda r^2}\mathbf{p} - \frac{e}{c}\mathbf{B}\times\left(\frac{\mathbf{r}}{\sqrt{1-\lambda r^2}} + \frac{\theta\times\sqrt{1-\lambda r^2}\mathbf{p}}{2\hbar}\right) \\ &im\omega\eta^0\left(\frac{\mathbf{r}}{\sqrt{1-\lambda r^2}} + \frac{\theta\times\sqrt{1-\lambda r^2}\mathbf{p}}{2\hbar}\right)\Bigg) + mc^2\Bigg]\widetilde{\Psi} = E\beta^0\widetilde{\Psi} \end{aligned} \tag{4.40}$$

where $\Psi(\mathbf{r},t) = e^{-\frac{iE_t}{\hbar}}\widetilde{\Psi}(\mathbf{r})$

4.4.1 Scalar particle case

The wave function is a vector with five components for a scalar particle of spin 0, which is written as follows.

$$\widetilde{\Psi}(\mathbf{r}) = \begin{pmatrix} \mathbf{\Phi} \\ i\psi \end{pmatrix} \text{ with } \mathbf{\Phi} \equiv \begin{pmatrix} \phi \\ \chi \end{pmatrix} \text{ and } \psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{4.41}$$

After replacing eq.(4.41) with eq.(4.40); we get the coupled system that follows.

$$mc^2\phi = E\chi + ic\mathbf{p}^{+}\cdot\psi, \tag{4.42}$$

$$mc^2\psi = ic\mathbf{p}^{-}\phi, \tag{4.43}$$

$$mc^2\chi = E\phi \tag{4.44}$$

with $\mathbf{p}^{+}$ and $\mathbf{p}^{-}$ defined in eq(4.3)

At this point, the problem can be solved directly by the two-dimensional deformed Klein-Gordon Oscillator in a constant magnetic field with the same differential equation (4.6) as the previous case in this chapter, when the system is uncoupled in favor of ϕ.

$$\left[(1-\lambda r^2)p^2+\eta_\theta^\lambda\frac{r^2}{1-\lambda r^2}+i\hbar\lambda(\mathbf{r}\cdot\mathbf{p})-\beta_\theta^\lambda L_z-\varepsilon_\theta^\lambda\right]\phi=0 \tag{4.45}$$

Hence, the energy spectrum could be such:

$$E_{n,l}^{\lambda,\theta}=\pm mc^2\left[1-\frac{2\omega\hbar}{mc^2}\Upsilon+\frac{2\hbar}{mc^2}\left(2n+l+1\right)\sqrt{\Phi\Gamma^\theta}+\frac{\lambda\hbar}{2m}\left(4n\left(n+l+1\right)\Gamma^\theta+2l+1\right)\right.$$
$$\left.-l\left(\widetilde{\omega}+\frac{m\left(\omega^2+\widetilde{\omega}^2\right)\theta}{2\hbar}+\frac{\omega}{2}\Upsilon\lambda\theta\right)\right]^{\frac{1}{2}} \tag{4.46}$$

And

$$\begin{aligned}\Gamma^\theta &= \left(\Upsilon+\frac{m^2\left(\omega^2+\widetilde{\omega}^2\right)\theta^2}{4\hbar^2}\right);\Upsilon=\left(1+\frac{m\widetilde{\omega}\theta}{\hbar}\right)\\ \Phi &= \left(\omega-\frac{\lambda\hbar}{2m}\right)^2+\widetilde{\omega}^2-\widetilde{\omega}\left(\omega-\frac{\lambda\hbar}{m}\right)\lambda\theta+\frac{\left(\omega^2+\widetilde{\omega}^2\right)\lambda^2\theta^2}{16}\end{aligned} \tag{4.47}$$

In terms of the variables r and φ, we can now write the general form of the wave function Ψ:

$$\Psi\left(r,\varphi\right)=C_n 2^{\frac{l}{2}}e^{il\varphi}\left(1-\lambda r^2\right)^{\frac{\mu}{2}}\left(\lambda r^2\right)^{\frac{l}{2}}P_n^{(l,\mu-1/2)}\left(1-2\lambda r^2\right) \tag{4.48}$$

4.4.2 vector case

We continue in the same manner as in the preceding one. In this case, the wave function of spin 1 is a vector with ten components noted by $\Psi\left(\mathbf{r}\right)^T=\left(i\varphi,\mathbf{A}\left(r\right),\mathbf{B}\left(r\right),\mathbf{C}\left(r\right)\right)$ with A_i, B_i and C_i $(i=1,2,3)$ being ,respectively the components of the vectors $\mathbf{A}\left(r\right),\mathbf{B}\left(r\right),\mathbf{C}\left(r\right)$.The equation (4.40) is reduced to the following system:

$$mc^2\varphi=-c\mathbf{p}^-\cdot\mathbf{B} \tag{4.49}$$

$$mc^2\mathbf{A}=E\mathbf{B}-c\mathbf{p}^+\times\mathbf{C} \tag{4.50}$$

$$mc^2\mathbf{B}=E\mathbf{A}+c\mathbf{p}^+\varphi \tag{4.51}$$

$$mc^2\mathbf{C} = -c\mathbf{p}^- \times \mathbf{A} \tag{4.52}$$

To decouple the system above , we eliminate φ,**B** and **C** in terms of **A** and we get:

$$\left(E^2 - m^2c^4\right)\mathbf{A} = c^2\mathbf{p}^+\left(\mathbf{p}^- \cdot \mathbf{A}\right) - c^2\mathbf{p}^+ \times \left(\mathbf{p}^- \times \mathbf{A}\right) - \frac{1}{m^2}\mathbf{p}^+\left[\mathbf{p}^- \cdot \left[\mathbf{p}^+ \times \left(\mathbf{p}^- \times \mathbf{A}\right)\right]\right] \tag{4.53}$$

which can be rewritten in the following form:

$$\left(E^2 - m^2c^4\right)\mathbf{A} = c^2\left[\left(\mathbf{p}^+ \cdot \mathbf{p}^-\right)\mathbf{A} - \left(\mathbf{p}^+ \times \mathbf{p}^-\right) \times \mathbf{A}\right] - \frac{1}{m^2}\mathbf{p}^+\left[\mathbf{p}^- \cdot \left[\mathbf{p}^+ \times \left(\mathbf{p}^- \times \mathbf{A}\right)\right]\right] \tag{4.54}$$

which can be rewritten in the following form: The evaluation of the first two. (4.54) terms gives by a direct calculation,

$$\begin{aligned}\left(\mathbf{p}^+ \cdot \mathbf{p}^-\right)\mathbf{A} \;=\; & \left[\left(\left(1 + \frac{m\widetilde{\omega}\theta}{2\hbar}\right)^2 + \frac{m^2\omega^2\theta^2}{4\hbar^2}\right)\left((1-\lambda r^2)p^2 + i\hbar\lambda(\mathbf{r}\cdot\mathbf{p})\right)\right. \\ & + \left(m^2\left(\omega^2 + \widetilde{\omega}^2\right) - m\omega\hbar\lambda\Upsilon\right)\frac{r^2}{1-\lambda r^2} - 2m\omega\hbar - \frac{m\omega eB\theta}{c} \\ & \left. - \left(\frac{eB}{c} + m\omega\lambda\theta + \frac{m^2\theta}{\hbar}\left(\omega^2 + \widetilde{\omega}^2\right) + \frac{eB\theta^2 m\omega}{4c\hbar}\lambda\right)L_z\right]\mathbf{A}\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}\left(\mathbf{p}^+ \times \mathbf{p}^-\right) \times \mathbf{A} \;=\; & \left[\left(\frac{eB}{c}\left(\lambda - \frac{2m\omega}{\hbar} + \frac{eB\theta\lambda}{4c\hbar}\right) + \frac{m^2\omega^2\theta}{\hbar}\lambda\right)\frac{r^2}{1-\lambda r^2}\right. \\ & - \left(m\omega\frac{\theta}{\hbar}\left(2 + \frac{eB\theta}{c\hbar}\right)(1+\lambda)\right)\left((1-\lambda r^2)p^2 + i\hbar\lambda(\mathbf{r}\cdot\mathbf{p})\right) \\ & \left. +2\xi L_z\right]S_z\mathbf{A}\end{aligned} \tag{4.56}$$

where

$$\xi = \left(\lambda + \frac{2m\omega}{\hbar} + \frac{eB\theta\lambda}{2c\hbar} + \frac{eB\theta m\omega}{c\hbar^2} + \frac{m^2\theta^2}{4\hbar^2}\lambda\left(\omega^2 + \widetilde{\omega}^2\right)\right) \tag{4.57}$$

After inserting both (4.55) and (4.56) into eq(4.54),and in order to explore the nonrelativistic limit of equation (4.54) despite the fact that there is no known analytical method to solve equations of this type. To do this, we treat the last term as unimportant as it is of order m^{-3}, which allows us to study the nonrelativistic limit.

This approximation causes the following equation 4.58 to resemble both (4.6and4.45) corresponding to the KG case and the scalar DKP particle, respectively. We discover

$$\begin{aligned}
\varepsilon'_{\lambda,\theta}\mathbf{A} &= \Bigg[\left(\left(\left(1+\frac{m\widetilde{\omega}\theta}{2\hbar}\right)^2+\frac{m^2\omega^2\theta^2}{4\hbar^2}+m\omega\frac{\theta}{\hbar}\left(2+\frac{eB\theta}{c\hbar}\right)(1+\lambda)\right)\left((1-\lambda r^2)p^2+i\hbar\lambda(\mathbf{r}\cdot\mathbf{p})\right)\right. \\
&+\left(\left(m^2\left(\omega^2+\widetilde{\omega}^2\right)-m\omega\hbar\lambda\Upsilon\right)-\left(\frac{eB}{c}\left(\lambda-\frac{2m\omega}{\hbar}+\frac{eB\theta\lambda}{4c\hbar}\right)+\frac{m^2\omega^2\theta}{\hbar}\lambda\right)S_z\right)\frac{r^2}{1-\lambda r^2} \\
&-\left(\frac{eB}{c}+m\omega\lambda\theta+\frac{m^2\theta}{\hbar}\left(\omega^2+\widetilde{\omega}^2\right)+\frac{eB\theta^2m\omega}{4c\hbar}\lambda+2\xi S_z\right)L_z\Bigg]\mathbf{A} \qquad (4.58)
\end{aligned}$$

where

$$\varepsilon'_{\theta,\lambda}=\varepsilon+2\left(\frac{eB}{c}+\frac{m^2\theta}{\hbar}\left(\omega^2+\widetilde{\omega}^2\right)\right)S_z+\frac{m\omega eB\theta}{c} \qquad (4.59)$$

By introducing the eigenvalues of S_z and L_z, we obtain the final expression of the spectrum energetic

$$\begin{aligned}
E_{nr} &= \Bigg[(2n+l+1)\,\hbar\sqrt{\left(\Phi\mp\left(4\widetilde{\omega}\left(\omega-\frac{\lambda\hbar}{2m}\right)+\frac{\left(\omega^2+\widetilde{\omega}^2\right)\lambda\theta}{4}\right)\right)\Gamma^{\lambda,\theta}} \\
&+\frac{\lambda\hbar^2}{2m}\left(4n\,(n+l+1)\,\Gamma^{\lambda,\theta}+2l+1\right)-\hbar l\left(\widetilde{\omega}+\frac{m\left(\omega^2+\widetilde{\omega}^2\right)\theta}{2\hbar}+\frac{\omega}{2}\Upsilon\lambda\theta\pm\Theta\right) \\
&-\left(\hbar\omega+m\omega\widetilde{\omega}\theta\pm\hbar\left(\widetilde{\omega}+\frac{m\theta\left(\omega^2+\widetilde{\omega}^2\right)}{2\hbar}\right)\right)\Bigg] \qquad (4.60)
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\lambda,\theta} &= \left(\Upsilon+\frac{m^2\left(\omega^2+\widetilde{\omega}^2\right)\theta^2}{4\hbar^2}\pm\left(\frac{2m\omega\theta}{\hbar}\left(1+\frac{m\widetilde{\omega}\theta}{\hbar}\right)(1+\lambda)\right)\right) \\
\Theta &= 2\left(\omega+\frac{\lambda\hbar}{2m}\right)+\widetilde{\omega}\lambda\theta+\frac{2m\widetilde{\omega}\omega\theta}{\hbar}+\frac{m\theta^2}{4\hbar}\lambda\left(\omega^2+\widetilde{\omega}^2\right) \qquad (4.61)
\end{aligned}$$

The outcome clearly shows the contributions of all the terms in eq.(4.61) and especially those due to the presence of both the spin and the deformations, as well as that of the additional spin-orbit terms in $2\xi S_z L_z$ which can be interpreted as due to the interaction between the two.

4.5 Thermodynamic Properties of KG and DKP in AdS and NC spaces

Let us focus on the thermodynamic properties of the system.The partition function at finite temperature T is:

$$Z=\sum_{n=0}^{\infty}e^{-\frac{E_n}{k_BT}}=\sum_{n=0}^{\infty}\exp\left(-\frac{mc^2}{k_BT}\sqrt{a_1+a_2n+a_3n^2}\right) \tag{4.62}$$

Here k_B is the Boltzmann constant and the expressions of the other parameters obtain from the energy spectrum eq(4.16):

$$\begin{aligned} a_1 &= 1+\frac{2\hbar}{mc^2}\left(\sqrt{\Phi\Gamma^\theta}+\frac{\lambda\hbar}{2m}-\omega\Upsilon\right) \\ a_2 &= \frac{4\hbar}{mc^2}\left(\sqrt{\Phi\Gamma^\theta}+\frac{\lambda\hbar}{m}\Gamma^\theta\right) \text{ and } a_3=\frac{4\lambda\hbar^2}{m^2c^2}\Gamma^\theta \end{aligned} \tag{4.63}$$

In order to evaluate this function, we use the Euler-MacLaurin formula:

$$\sum_{n=0}^{\infty}f(n)=\frac{1}{2}f(0)+\int f(x)\,dx-\sum_{p=1}^{\infty}\frac{1}{(2p)!}B_{2p}f^{(2p-1)}(0) \tag{4.64}$$

B_{2p} are the Bernoulli numbers, $f^{(2p-1)}$ is the derivative of order $(2p-1)$ and the integral term is given by:

$$\begin{aligned} I &= \frac{2a_1}{\sqrt{a_2^2-4a_1a_3}}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\left(\frac{4a_1a_3}{a_2^2-4a_1a_3}\right)^n \\ &\times\left[\frac{\Gamma(2n+2)}{\chi^{2n+2}}-\frac{e^{-\chi}}{(2n+2)}\Phi(1,2n+2,\chi)\right] \end{aligned} \tag{4.65}$$

where we have used $\chi=\frac{mc^2}{k_BT}\sqrt{a_1}$, the new variable $y=\sqrt{1+\frac{a_2}{a_1}n+\frac{a_3}{a_1}n^2}$and the power series of the square root of the integral:

$$I=\frac{2a_1}{\sqrt{a_2^2-4a_1a_3}}\int_1^{+\infty}\exp(-\chi y)\left(1+\frac{4a_1a_3}{a_2^2-4a_1a_3}y^2\right)ydy \tag{4.66}$$

At high temperatures, we can ignore the first and the third terms in eq.(4.64) and keep only the integral. Similarly, we neglect the $e^{-\chi}$ term beside the $\chi^{-(2n+2)}$ one in eq.(4.65). Therefore, the partition function becomes:

$$Z=\left(\frac{k_BT}{mc^2}\right)^2\frac{2}{\sqrt{a_2^2-4a_1a_3}}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\Gamma(2n+2)\sigma^n \tag{4.67}$$

with:

$$\sigma=\left(\frac{k_BT}{mc^2}\right)^2\frac{4a_3}{\left(a_2^2-4a_1a_3\right)} \tag{4.68}$$

We restrict ourselves to the first order in λ,and we neglect the $(k_BT)^{-2}$ term in parentheses and we obtain the high-temperature expansion of the partition function:

$$Z\simeq\frac{(k_BT)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2+\omega^2}}\left(1-\frac{3(k_BT)^2\lambda}{m^2c^2\left(\widetilde{\omega}^2+\omega^2\right)}-\frac{m\widetilde{\omega}}{2\hbar}\theta\right) \tag{4.69}$$

The first term is the usual partition function for a 2D scalar bosonic oscillator with a uniform magnetic field in the r-representation. The second and third terms express the contribution that comes from the space deformation through the AdS and NC algebras

At this stage, we can evaluate all the thermodynamic properties of our system (free energy F, mean energy U, specific heat C and entropy S) using their definitions [37]:

$$F=-k_BT\ln Z, U=k_BT^2\frac{\partial \ln Z}{\partial T}, C=\frac{\partial U}{\partial T} \text{ and } S=-\frac{\partial F}{\partial T} \tag{4.70}$$

$$F=-k_BT\ln\left(\frac{(k_BT)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2+\omega^2}}\left(1-\theta\,(k_BT)^2-\varkappa\right)\right) \tag{4.71}$$

$$U=4k_BT\left[1-\frac{1}{2\left(1-\theta\,(k_BT)^2-\varkappa\right)}\right] \tag{4.72}$$

$$C=4k_B\left[1-\frac{1+\theta\,(k_BT)^2-\varkappa}{2\left(1-\theta\,(k_BT)^2-\varkappa\right)^2}\right] \tag{4.73}$$

$$S=k_B\left[\frac{2-4\left(\theta\,(k_BT)^2-\varkappa\right)}{1-\theta\,(k_BT)^2-\varkappa}+\ln\left(\frac{(k_BT)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2+\omega^2}}\left(1-\theta\,(k_BT)^2-\varkappa\right)\right)\right] \tag{4.74}$$

where we have used the parameters $\theta=\frac{3\lambda}{m^2c^2\left(\widetilde{\omega}^2+\omega^2\right)}, \varkappa=\frac{m\widetilde{\omega}}{2\hbar}\theta$

We can simply check these expressions in different manners. We use the limit $\widetilde{\omega}\to 0$ (i.e. $B\to 0$) and $\varkappa=0$, we obtain the thermodynamic results of the deformed 2D scalar bosonic oscillator with AdS commutation relations[72].If we use the limit $\lambda=\theta=0$, we obtain the thermal properties of the ordinary 2D bosonic oscillator for both KG and scalar DKP particles in a uniform magnetic field.

We show in figures 4.1 to4.4, the dependence of these thermodynamic properties with the temperature T for different values of the deformation parameters $\lambda$and θ. We chose

$\omega = B = 1$ and we use the Hartree atomic units ($\hbar = c = k_B = m = 1$).

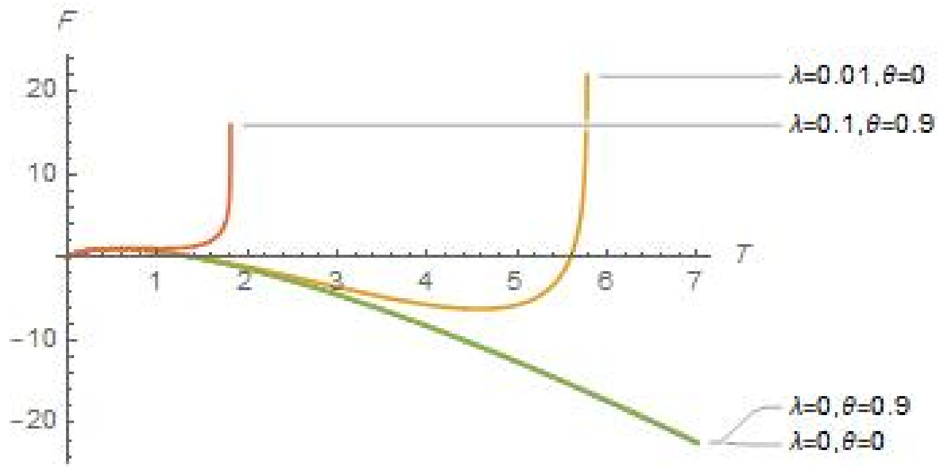


Figure.4.1 The free energy F function of T for different values of the AdS and NC parameters λ, θ.

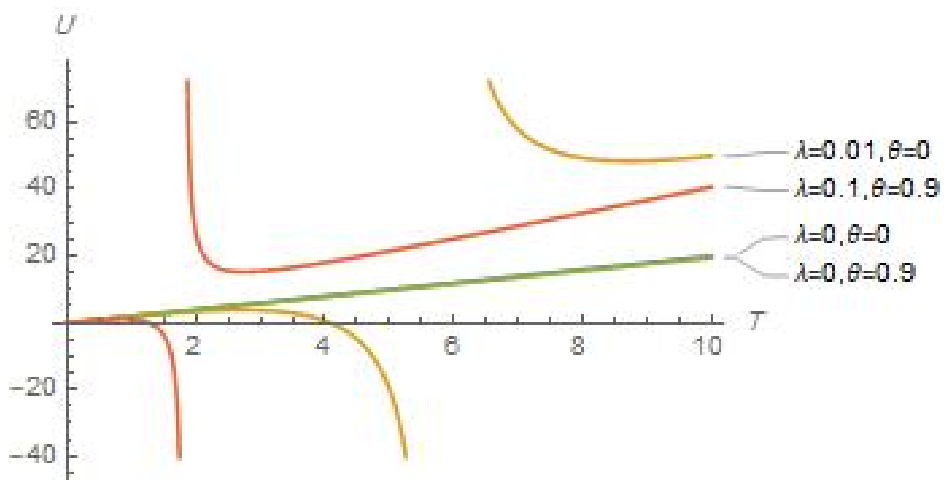


Figure.4.2 The mean energy U function of T for different values of the AdS and NC parameters λ, θ

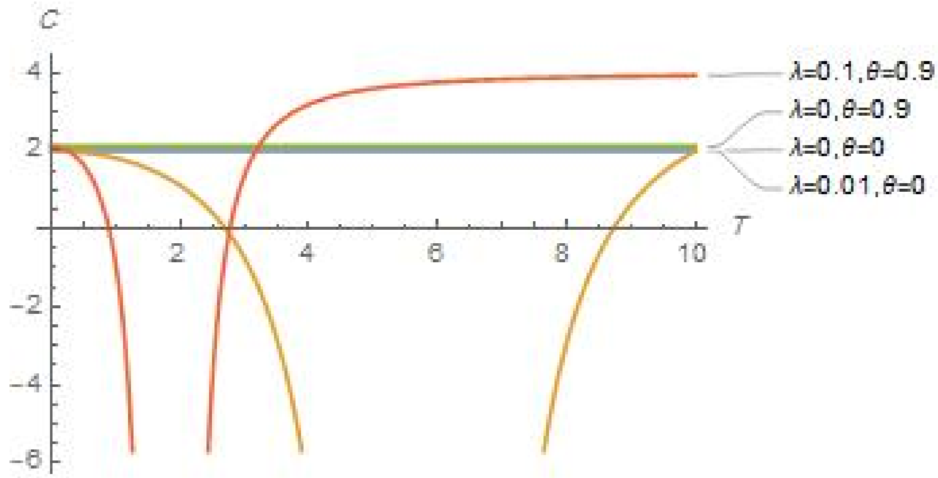


Figure.4.3 The specific heat C function of T for different values of the AdS and NC parameters λ, θ

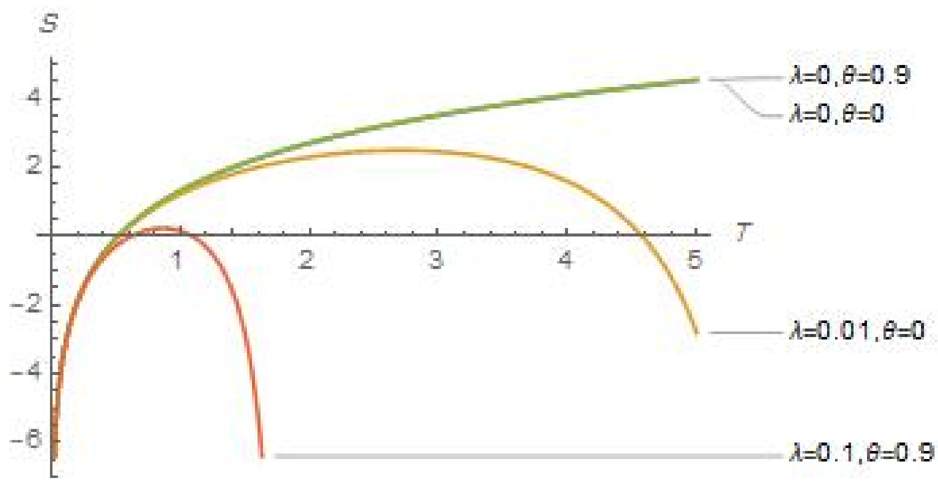


Figure.4.4 The entropy S function of T for different values of the AdS and NC parameters λ, θ

We clearly remark that the influence of the NC space deformation has a smallness effect on all the curves.

Fig.4.1, for the free energy,shows that it grows rapidly in the deformed cases $(\lambda, \theta \neq 0)$ unlike the normal case where it continues its decrease to infinity; this after a small increase followed by a decrease common to both cases. For a fixed value of T , the free energy increases with the deformation parameter λ.

Fig.4.2 shows that there is a discontinuity in the mean energy U at a certain temperature $T_c = \sqrt{\frac{1+\theta\frac{\sqrt{\alpha}}{2}}{3\lambda}}$ where it decreases rapidly in the vicinity of T_c and then it grows slightly as in the ordinary case.

In Fig.4.3, we see that the specific heat C decreases until the discontinuity point T_c, then it increases. On the other hand, it is constant for $\lambda, \theta = 0$ (namely, $C = 2k_B$).

For the entropy function S, we see in Fig.4.4 that it grows to a maximum value then decreases to infinity and this contrasts with the ordinary case where it has a continuous growth with T

4.6 Thermodynamic Properties of Dirac in AdS and NC spaces

Let us focus on the thermodynamic properties of the DO in AdS and NC spaces system.The partition function at finite temperature T is:

$$Z = \sum_{n=0}^{\infty} e^{-\frac{E_n}{k_B T}} = \sum_{n=0}^{\infty} \exp\left(-\frac{mc^2}{k_B T}\sqrt{a_1 + a_2 n + a_3 n^2}\right) \tag{4.75}$$

Here k_B is the Boltzmann constant and the expressions of the other parameters obtain from the energy spectrum eq(4.34):

$$a_1 = 1 + \frac{2\hbar}{mc^2}\left(\sqrt{\left(\Phi + \left(2\widetilde{\omega}\left(\omega - \frac{\lambda\hbar}{2m} - \frac{\widetilde{\omega}\theta\lambda}{4}\right) - \frac{\omega^2\theta}{2}\lambda\right)\tau\right)\Omega^\tau} + \frac{\lambda\hbar}{2m} - \omega\Upsilon\right)$$
$$a_2 = \frac{4\hbar}{mc^2}\left(\sqrt{\left(\Phi + \left(2\widetilde{\omega}\left(\omega - \frac{\lambda\hbar}{2m} - \frac{\widetilde{\omega}\theta\lambda}{4}\right) - \frac{\omega^2\theta}{2}\lambda\right)\tau\right)\Omega^\tau} + \frac{\lambda\hbar}{m}\Omega^\tau\right) \text{ and } a_3 = \frac{4\lambda\hbar^2}{m^2c^2}\Omega^\tau \tag{4.76}$$

We restrict ourselves to the first order in λ,and we neglect the $(k_B T)^{-2}$ term in parentheses and we obtain the high-temperature expansion of the partition function:

$$Z \simeq \frac{(k_B T)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2 + \omega^2 + 2\omega\widetilde{\omega}\tau}}\left(1 - \frac{3\,(k_B T)^2\,\lambda}{m^2c^2\left(\widetilde{\omega}^2 + \omega^2 + 2\omega\widetilde{\omega}\tau\right)} - \frac{m\left(\widetilde{\omega} + \omega\tau\right)}{2\hbar}\theta\right) \tag{4.77}$$

The first term is the usual partition function for a 2D fermionic oscillator with a uniform magnetic field in the r-representation. The second and third terms express the contribution that comes from the space deformation through the AdS and NC algebras

At this stage, we can evaluate all the thermodynamic properties of our system (free energy F, mean energy U, specific heat C and entropy S) using their definitions [37]:

$$F=-k_BT\ln Z, U=k_BT^2\frac{\partial\ln Z}{\partial T}, C=\frac{\partial U}{\partial T} \text{ and } S=-\frac{\partial F}{\partial T} \tag{4.78}$$

$$F=-k_BT\ln\left(\frac{(k_BT)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2+\omega^2+2\omega\widetilde{\omega}\tau}}\left(1-\theta^\tau(k_BT)^2-\varkappa^\tau\right)\right) \tag{4.79}$$

$$U=4k_BT\left[1-\frac{1}{2\left(1-\theta^\tau(k_BT)^2-\varkappa^\tau\right)}\right] \tag{4.80}$$

$$C=4k_B\left[1-\frac{1+\theta^\tau(k_BT)^2-\varkappa^\tau}{2\left(1-\theta^\tau(k_BT)^2-\varkappa^\tau\right)^2}\right] \tag{4.81}$$

$$S=k_B\left[\frac{2-4\left(\theta^\tau(k_BT)^2-\varkappa^\tau\right)}{1-\theta^\tau(k_BT)^2-\varkappa^\tau}+\ln\left(\frac{(k_BT)^2}{2\hbar mc^2\sqrt{\widetilde{\omega}^2+\omega^2\omega\widetilde{\omega}\tau}}\left(1-\theta^\tau(k_BT)^2-\varkappa^\tau\right)\right)\right] \tag{4.82}$$

where we have used the parameters $\theta^\tau=\frac{3\lambda}{m^2c^2\left(\widetilde{\omega}^2+\omega^2+2\widetilde{\omega}\omega\right)}, \varkappa^\tau=\frac{m(\widetilde{\omega}+\omega\tau)}{2\hbar}$

We can simply test these expressions in different manners. We use the limit $\widetilde{\omega}\to 0$ (i.e. $B\to 0$) and $\theta=0$, we obtain the thermodynamic results of the deformed 2D fermionic oscillator with AdS commutation relations.If we use the limit $\lambda,\theta\to 0$, we obtain the thermal properties of the ordinary 2D DO in a uniform magnetic field.

We show in figures 4.5 to4.12, the dependence of these thermodynamic properties with the temperature T for different values of the deformation parameters $\lambda$and θ. We chose $\omega=B=1$ and we use the Hartree atomic units ($\hbar=c=k_B=m=1$).

•For $\tau = +1$

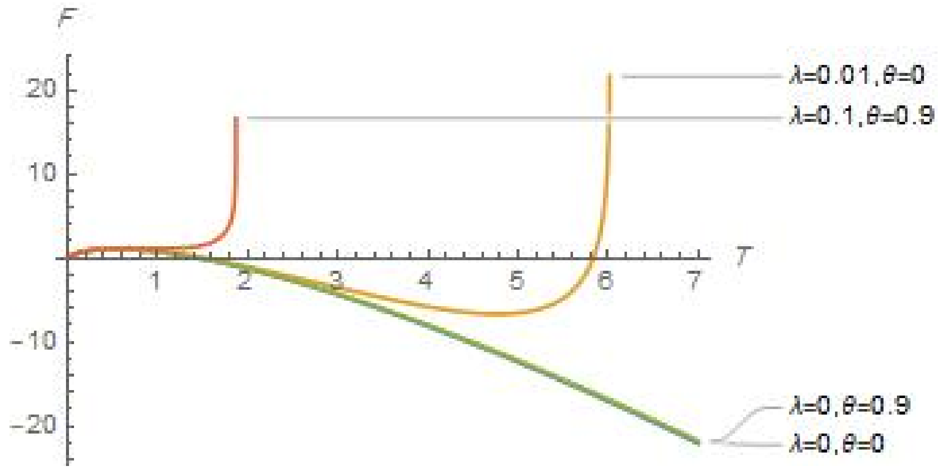


Figure.4.5 The free energy F function of T for different values of the AdS and NC parameters λ, θ for $\tau = +1$.

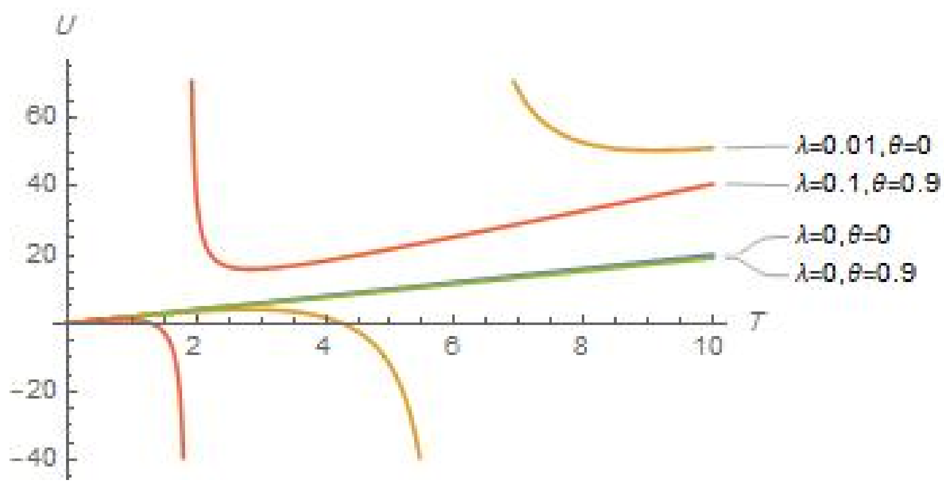


Figure.4.6The mean energy U function of T for different values of the AdS and NC parameters λ, θ for $\tau = +1$.

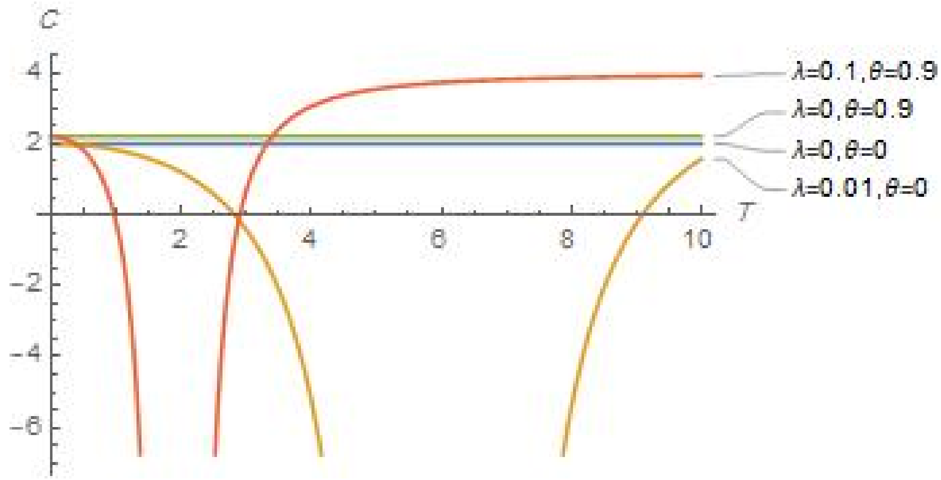


Figure.4.7 The specific heat C function of T for different values of the AdS and NC parameters λ, θ for $\tau = +1$.

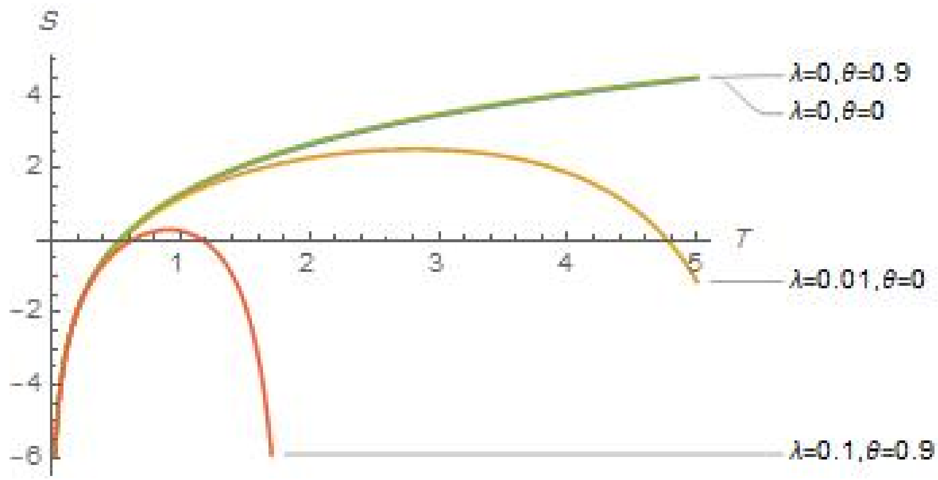
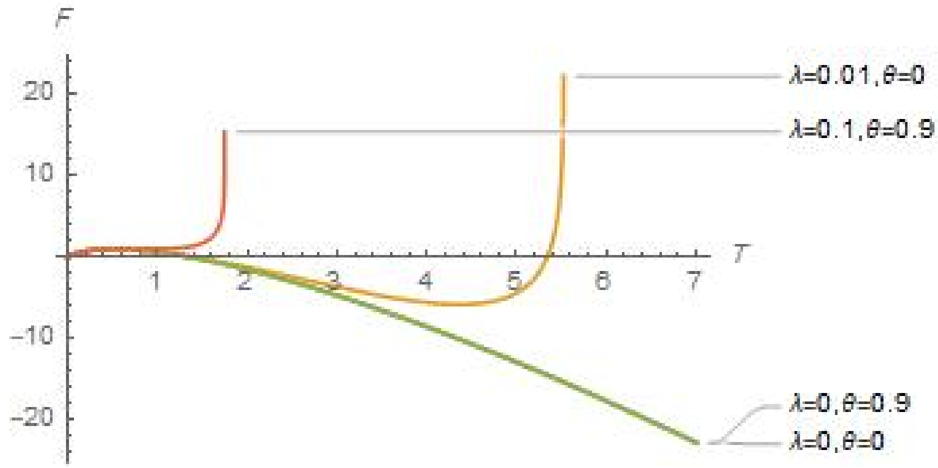
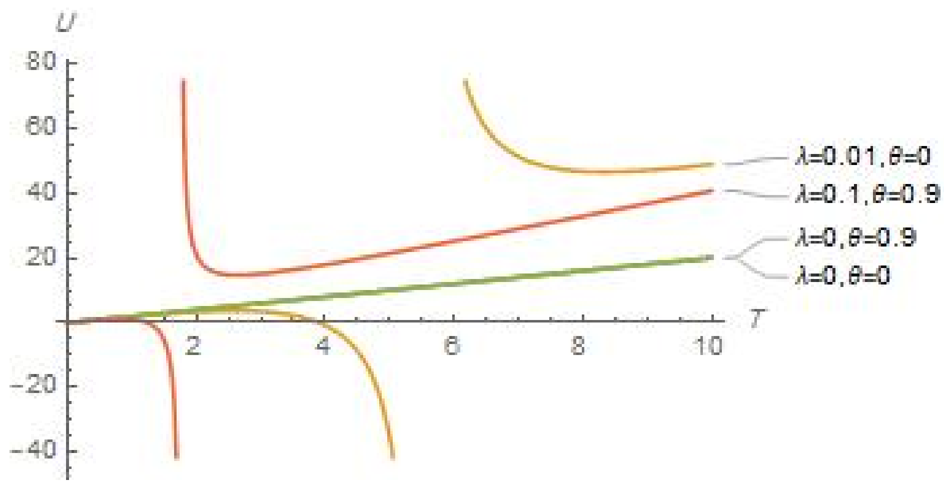


Figure.4.8 The entropy S function of T for different values of the AdS and NC parameters λ, θ for $\tau = +1$.

•For $\tau = -1$



$Figure$.4.9 The free energy F function of T for different values of the AdS and NC parameters λ, θ for $\tau = -1$.



$Figure$.4.10 The mean energy U function of T for different values of the AdS and NC parameters λ, θ for $\tau = -1$.

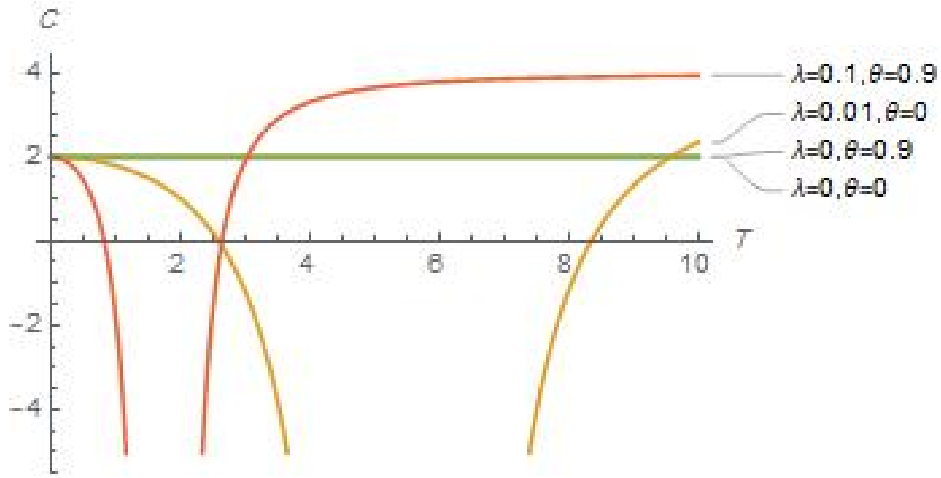


Figure.4.11 The capacity heat C function of T for different values of the AdS and NC parameters λ, θ for $\tau = -1$.

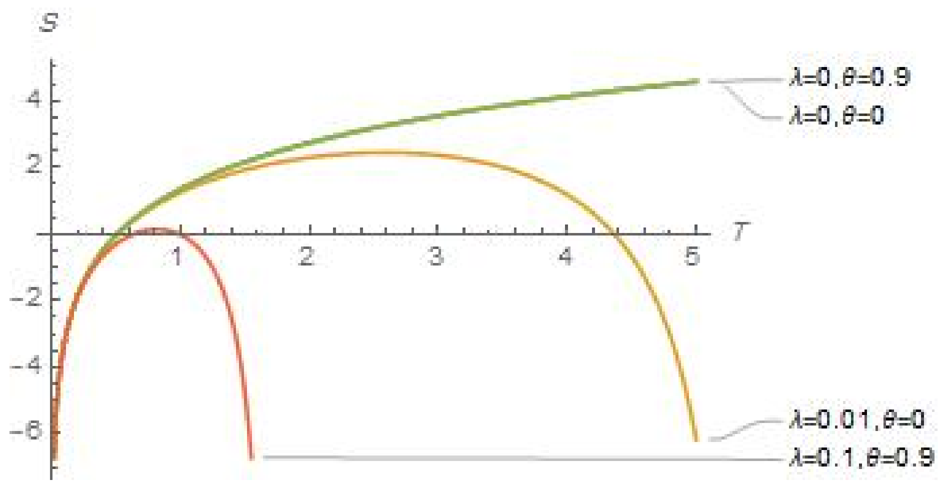


Figure.4.12 The entropy S function of T for different values of the AdS and NC parameters λ, θ for $\tau = -1$.

We obviously notice that the influence of the NC space deformation has a smallness effect on all the curves.

Figs.(4.5-4.9), for the free energy,shows that it grows rapidly in the deformed cases $(\lambda, \theta \neq 0)$ unlike the normal case where it continues its decrease to infinity; this after a small increase followed by a decrease common to both cases. For a certain value of T , the free energy increases with the deformation parameter λ.

Figs.(4.6-4.10) shows that there is a discontinuity in the mean energy U at a certain temperature $T_c = \sqrt{\frac{1+\theta\frac{\sqrt{\alpha}}{2}}{3\lambda}}$ where it decreases rapidly in the vicinity of T_c and then it grows slightly as in the ordinary case.

In Figs.(4.7-4.11), we see that the specific heat C decreases until the discontinuity point T_c, then it increases. On the other hand, it is constant for $\lambda, \theta = 0$ (namely, $C = 2k_B$).

For the entropy function S, we see in Figs.(4.8-4.12) that it grows to a maximum value then decreases to infinity and this contrasts with the ordinary case where it has a continuous growth with T

General Conclusion

In this thesis, we mainly studied the relativistic quantum mechanics of Klein Gordon, Dirac, and Duffin Kemmer Petiau equations following two different deformation formalisms.

In the first chapter, we have presented the three-based equations and their properties in quantum mechanics.

In the second chapter, we investigated the exact solutions of the KG, Dirac, and DKP (scalar and vector cases) equations for the 2D oscillator with an external uniform magnetic field within the framework of deformed quantum mechanics with anti-de Sitter commutation relations. The minimal uncertainty in the measurement of the momentum is nonzero as a result of the AdS deformations. By using the Nikiforov-Uvarov method, we were able to derive analytical expressions for the bound state energies and the system s wave functions. With additional corrections based on the deformation parameter Z, we analytically expressed the system s eigenfunctions in terms of the Jacobi polynomials and the corresponding eigen-energies in each case. Our findings demonstrate that even for large values of the principal quantum number, the deformed spectrum maintains its discrete nature, and the EUP de- formation thus eliminates the degeneracy of the spectrum found in the conventional case (without deformations). We were able to determine an experimental limit deformation parameter using this. We focused our study on the nonrelativistic limit of the vector DKP oscillator because it was nearly impossible to come up with an exact solution to the problem. In this instance, the spin s nonzero value led to the presence of corrective terms in addition to those resulting from the deformation. Thus, we were able to obtain the corrective terms that represented the effects of the spin and its interactions with the magnetic field(which is similar as in the literature), with the orbital moment (the typical spin-orbit term), and with a new "spin-orbit" term that represented the interaction of the spin with the orbital moment and the space deformation at the same time. A contribution proportional to the deformation parameter λ is made to the spin-orbit term by the EUP as a result of this new interaction.

We also looked at how our system's thermodynamic characteristics were impacted by the space deformation. The graphical displays of our findings demonstrated that the effects of EUP on the statistical properties are significant only in the high temperature regime; these findings are consistent with those for graphene in Snyder-de Sitter curved space.[73] This difference is more pronounced for the free energy, which rises in the deformed case while falling in the ordinary case, and the entropy, which decreases when there is deformation present but increases when there is no more deformation.

In the third chapter, we generalized this same relativistic equations in noncommutative space, dealing with the problem of the oscillator of KG,Dirac and DKP with the interaction of a magnetic field given by the non-minimal substitution in noncommutative space.The calculation was performed by a direct method for all cases of and was compared with the cases in ordinary space. For the case of spin 1 in DKP equation, we found that the problem is equivalent to the case of a DKP vector boson moving in a constant magnetic field with an additional term that depends on the parameter of noncommutativity.According to thorough and precise calculations, the Landau problem in commutative space and the relativistic oscillator in a uniform magnetic field exhibit behaviors that are similar to each other.

We then used the Euler-MacLaurin method to investigate the system's thermodynamic characteristics, and we visualized our numerical findings by evaluating the thermodynamic functions using the associated partition function Z. Additionally, the impact of the NC parameter on thermodynamic characteristics was covered. We demonstrate an explicit dependence between the energy spectrum and related thermodynamic functions on the NC parameter θ, which describes the noncommutativity of the space. It is also noted that even though the NC parameter is small, the impact of NC space on the thermodynamic properties cannot be ignored.

In fourth chapter, the KGO,DO,and DKPO in two dimensions with an external magnetic field has been precisely solved using relativistic quantum mechanics with AdS and in the NC space.We compared our result to the previous chapters and discussed the special cases that arise. It demonstrated that for the AdS parameter and the NC space parameter, the energy eigenvalues increase monotonically with the magnetic field variable, and for one principal quantum number, the energy increases with the increase of the azimuthal quantum number.

In the end, we analyzed the thermodynamic properties of each system in both spaces at high temperature regime,according to the results, we found that the systems have been affected by the deformations of spaces and we remarked that the influence of AdS is considerable than the NC in all findings.

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Abstract

In this thesis, we conducted an analytical study at the atomic scale of two-dimensional relativistic deformed bosonic and fermionic oscillator equations for charged particles subject to the effect of a uniform magnetic field. In the first stage, we consider the presence of a minimal uncertainty in momentum caused by the anti-deSitter space model and we use the Nikiforov Uvarov method (NU) to solve the system. The exact energy eigenvalues and the corresponding wave functions are obtained using Jacobi polynomials, and we find that the deformed spectrum remains discrete even for large values of the principal quantum number. In addition, after evaluating the thermal properties, we find that they have been affected by the deformation of the space at high-temperature regime. In the second stage, we solve the same equations in the non-commutative space using a direct method to obtain the energy spectrums and the wave functions, we detect that the study has similar behaviors to the Landau problem in commutative space. Then, the outcomes of the thermal properties show that the systems have been influenced by the NC space. At the last stage, we generalize both deformations in the relativistic equations, and we solve the systems using the NU method hence we obtained the exact energy spectrums and the wave functions by applying the Jacobi polynomials; in the end, we examine the thermal properties which have been influenced by the two deformations.

Résumé

Dans cette thèse, nous avons mené une étude analytique au niveau atomique des équations d'oscillateur relativiste bosonique et fermionique déformées bidimensionnelles pour des particules chargées soumises à l'effet d'un champ magnétique uniforme. Dans la première étape, nous considérons la présence d'une incertitude minimale sur la quantité de mouvement causée par le modèle spatial anti-deSitter et nous utilisons la méthode Nikiforov Uvarov (NU) pour résoudre les systèmes. Les valeurs propres exactes de l'énergie et les fonctions d'onde correspondantes sont obtenues à l'aide de polynômes de Jacobi, et nous constatons que le spectre déformé reste discret même pour de grandes valeurs du nombre quantique principal. De plus, après évaluation des propriétés thermiques, nous constatons qu'elles ont été affectées par la déformation de l'espace au régime de haute température. Dans la deuxième étape, nous résolvons les mêmes équations dans l'espace non commutatif en utilisant une méthode directe pour obtenir les spectres d'énergie et les fonctions d'onde, nous détectons que l'étude à des comportements similaires au problème de Landau dans l'espace commutatif. Ensuite, les résultats des propriétés thermiques montrent que les systèmes ont été influencés par l'espace NC. À la dernière étape, nous généralisons les deux déformations dans les équations relativistes, et nous résolvons les systèmes en utilisant la méthode NU, nous avons donc obtenu les spectres d'énergie exacts et les fonctions d'onde en appliquant les polynômes de Jacobi. En fin, nous examinons les propriétés thermiques qui ont été influencés par les deux déformations.

ملخص

في هذه الأطروحة، أجرينا دراسة تحليلية على المستوى الذري لمعادلات نسبية لهزاز بوزوني وفرميوني مشوهة ثنائية الأبعاد للجسيمات المشحونة الخاضعة لتأثير مجال مغناطيسي منتظم. في المرحلة الأولى، نأخذ في الاعتبار وجود حد أدنى من عدم اليقين في الزخم الناجم عن نموذج الفضاء المضاد ديسيتر ونستخدم طريقة Nikiforov Uvarov لحل الأنظمة. يتم الحصول على القيم الدقيقة للطاقة والدوال الموجية المقابلة باستخدام Jacobi متعدد الحدود، ونجد أن الطيف المشوه يظل منفصلاً حتى بالنسبة للقيم الكبيرة للرقم الكمي الرئيسي. بالإضافة إلى ذلك، بعد تقييم الخصائص الحرارية، وجدنا أنها تأثرت بتشوه الفضاء في نظام درجات الحرارة العالية. في المرحلة الثانية، قمنا بحل المعادلات نفسها في الفضاء غير التبديلي باستخدام طريقة مباشرة للحصول على أطياف الطاقة والوظائف الموجية، اكتشفنا أن الدراسة لها سلوكيات مماثلة لمشكلة لانداو في الفضاء تبديلي. بعد ذلك، تظهر نتائج الخصائص الحرارية أن الأنظمة قد تأثرت بالفضاء المدروس NC. في المرحلة الأخيرة، قمنا بتعميم كل من التشوهين في المعادلات النسبية، وقمنا بحل الأنظمة باستخدام طريقة NU ومن ثم حصلنا على أطياف الطاقة الدقيقة ودوال الموجة من خلال تطبيق جاكوبي كثيرات الحدود، وفي النهاية، قمنا بفحص الخصائص الحرارية التي قد تأثرت بالتشوهين الناتجين من الفضاء.