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Stability Analysis of Delay Differential Equation

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Dedicace

I dedicate the fruit of my humble effort

to the one who gave me life and hope,

and growing up with a passion for knowledge and knowledge, out of righteousness, kindness, and loyalty to them: my dear father, Kamal, and my dear mother, Naima. To those whom God gave me the blessing of their presence in my life, my brothers Yahya and Abdessalam,

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Abstract

The study of delay differential equations (DDEs) aims to understand the behavior of systems where the current state depends not only on present conditions but also on past states. The main objective of this work is to discuss the existence and uniqueness of solutions and give an analytic method to solve DDEs. Furthermore, we aim to study the stability of both linear and nonlinear equations to determine the conditions under which the system remains stable. The analysis will include examining equilibrium points and characteristic equations.

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Introduction

In nature, many changes take time to complete and do not happen immediately. For example, in biology, many processes do not occur immediately, such as human birth. The birth rate depends on available resources at some point. These cases cannot be used using the ordinary differential equation (ODE). So delay differential equations (DDE) were introduced to create more realistic models, and since then many processes have relied on past history.

Delay differential equations have a rich history and have found numerous applications in various fields The development of numerical methods for solving (DDEs) has made it possible to simulate and analyze complex systems that exhibit time delays (DDEs) continue to be an active area of research, with new applications and theoretical results being discovered all the time. During the 1908 International Congress of Mathematicians, Picard underscored the importance of incorporating hereditary effects into models of physical systems. Volterra's 1931 book laid the groundwork for understanding how hereditary effects influence species interaction models. DDEs gained momentum post-1940, driven by engineering and control challenges. Significant activity in the 1950s, highlighted by Myshkis (1951) and Krasovskii (1959), furthered DDE research. By the 1960s, Bellman and Cooke (1963) and Halanay (1966) provided comprehensive insights into the subject, marking a clear progression up to the early 1960s. Differential Delayed Equations (DDEs) is a type of differential equation where the derivative of the unknown function is given at a given time by indication of the values of the function at previous times. (DDEs) is also called time delay systems, systems with subsequent effect, genetic systems or differential equations. It belongs to the category of functional status systems, and (DDEs) has been widely taken in modeling physical and biological phenomena that show time delays in their dynamics. For example, (DDEs) is widely used to model the dynamics of time delay groups in their spread, the spread of infectious diseases with incubation periods, and the synchronization of oscillations associated with late interactions.

The dynamical behavior of differential equation systems with delay will be considered in this dissertation. Our work is divided into two chapters. We start with some results on DDEs, and the second chapter presents their stability analysis. The content of each chapter is outlined as follows:

In the first chapter we present a few fundamental ideas and essential findings, including the definition of differential equation systems with delay, existence and uniqueness of solutions, analytical solution via the method of steps and the use of several instances.

In the second chapter we explore the stability of the equilibrium points in both linear and non-linear case, and we conclude this chapter by the study of two systems in dimensions 1 and 2, using numerical simulations.[6], [12]

Chapter 1

General results on delay differential equations

This chapter includes some basic concepts and some necessary results, such as defining systems of differential equations with delay, the existence and uniqueness of solution, and the analytical solution by giving some illustrative examples.

Definition 1.1 [5] *A delay differential equation (DDEs) is a class of differential equations that involve delays or memory effects in their formulations in which the derivative of the solution depends on the state at the present time t. and on the state at earlier times.*

We will be working with (DDEs) of the form :

$$x'(t) = f(t, x(t), x(t-\tau)) \qquad t \geq t_0,\ x \in \mathbb{R}^n, \tag{1.1}$$

where $\tau > 0$ is the delay term.

DDEs have also been used in a wide variety of fields, such as physics, chemistry, biology, economy and neuroscience, to model various phenomena that exhibit time delays in their dynamics. There are different kinds of differential equations delay in which the delay is in different forms :

Definition 1.2 (Types of DDEs) [5] *:*

1. *We call delay differential equation with discrete delays (or constant delays), if the delay τ is a positive real number, or a brunch of delays*

$$x'(t) = f(t, x(t), x(t-\tau_1), ..., x(t-\tau_m)) \qquad t \geq t_0, x \in \mathbb{R}^n,$$

with all $\tau_j\,(j = 1, ..., m)$ being positive real numbers and $f : \mathbb{R} \times \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ is a nonlinear smooth function.

2. *We call delay differential equation with time-dependent delays, if τ depends on time $\tau = \tau(t)$, where delay $\tau(t) > 0$ is a given function.*

3. *We call delay differential equation with state dependent delays, if τ depends on $x(t)$; $\tau = \tau(t, x(t))$, where delay $\tau(t, x(t)) > 0$ depends on solution.*

There are other types of DDEs (such DDEs with distributed delays, DDEs of neutral type, etc...).

Example 1.1 (Mackey-Glass equation) *A model of circulating white blood cell numbers*

$$x'(t) = -\gamma x(t) + \beta \frac{x(t-\tau)}{1 + x(t-\tau)^n}, \qquad x(t) \in \mathbb{R}.$$

Example 1.2 (Pantograph equation) *Originates from modelling pantographs*

$$x'(t) = ax(t) + bx(kt), \qquad x(t) \in \mathbb{R}^n,$$

where a, b and k are parameters with $k \in \,]0, 1[$.

Example 1.3 (Sawtooth equation) *A model problem introduced by Mallet-Paret and Nussbaum*

$$\epsilon x'(t) = -\gamma x(t) - kx(t - a - cx(t)), \qquad x(t) \in \mathbb{R},$$

with $\epsilon, a, c > 0$ and $\gamma + k > 0$. This model gets its name from stable period solutions seen in $\epsilon \to 0$ limit.

In this work we will focus on the first case, which is the differential equation with one delay.

1.1 Existence and uniqueness of an initial value problem (IVP) of a DDE

Example 1.4 (Delayed malthusian) [6] *The familiar Malthusian Model describing the growth of a single population is given by*

$$\frac{dx}{dt} = rx(t), \tag{1.2}$$

where $r > 0$ is the growth rate. This model predicts exponential growth or exponential decline. To account for the influence of the past on the present population, we consider the following DDE:

$$\frac{dx}{dt} = rx(t - \tau). \tag{1.3}$$

1. *The term $rx(t-\tau)$ represents the population growth rat at time t, which depends on the population size $x(t-\tau)$ at a previous time $t-\tau$.*

2. *Here τ is the delay and it accounts for the time it takes for changes in resource availability to affect population growth.*

By integrating (1.3), we obtain the following integral equation :

$$x(t) = x(t_0) + \int_{t_0}^{t} rx(s-\tau)\,ds$$

$$x(t) = x(t_0) + \int_{t_0-\tau}^{t-\tau} rx(s)ds. \tag{1.4}$$

This integral representation suggests that knowing the value of $x(t_0)$ is not enough to calculate the values of $x(t)$ for $t > t_0$, also we must know the values $x(t)$ for $t \in [t_0 - \tau, t_0]$.

Therefore the kind of initial conditions that should be used in DDEs differ from ODEs so that one should specify in DDEs an initial function on some interval of length τ, say $[t_0 - \tau, t_0]$ and then try to find the solution of equation (1.1) for all $t \geq t_0$.

Definition 1.3 (History function) *The history function in the delayed differential equations (1.1) represents the values of the function at previous points of time*

$$x(t) = \varphi(t), \qquad t_0 - \tau \leq t \leq t_0,$$

which provides the information required to calculate the derivative at the present time. The history function determines the initial conditions of the delayed differential equations.

Definition 1.4 (The initial value problem) *The problem for finding a solution of the delay differential equation (1.1) with single delay $\tau > 0$*

$$x'(t) = f(t, x(t), x(t-\tau)) \quad \text{for} \quad t \geq t_0, \tag{1.5}$$

satisfiying the initial condition

$$x(t) = \varphi(t) \quad \text{for} \quad t \in J_- = [t_0 - \tau, t_0], \tag{1.6}$$

is called the initial value problem.

Theorem 1.1 *Let $\tau > 0$ be a constant in $J = [t_0, t_0 + a]$, where $t_0 \geq 0$, and $a > 0$. Let's consider the initial value problem (1.5), (1.6). Assume that $f(t, x, y)$ and $f_x(t, x, y)$, $f_y(t, x, y)$ are continuous on $\mathbb{R} \times \mathbb{R}^{2n}$ and φ is a given continuous function on $\mathbb{R}$. Then the initial value problem (1.5), (1.6) has exactly one solution.*

1.2 Solutions of delay differential equations

In this section, we introduce an elementary method that can be used to solve some delay differential equations analytically. It is called the method of steps, which converts the DDE on a given interval to an ODE over that interval, by using the known history function for that interval.

1.2.1 Principle of method of steps:

The method of steps used to solve delayed differential equations, where the late differential equation is converted into the ordinary differential equation within small intervals of time and each interval the solution at present depends on the solution in previous times.

[6] We will work with the differential equations of the continuous delay specified for $t > t_0$ and the history function on the interval $[t_0 - \tau, t_0]$ where τ is the delay.

- First, we will reduce the delayed differential equation (DDE) to the ordinary differential equation (ODE)in the interval $[t_0 - \tau, t_0]$.

- Then we find a right solution in this interval $[t_0 - \tau, t_0]$ and we will use it as a initial function in this interval $[t_0 + \tau, t_0 + 2\tau]$.

- Then we find a solution in the interval $[t_0 + \tau, t_0 + 2\tau]$ and so we continue the same way from the interval $[t_0 + \tau, t_0 + 2\tau]$ to the interval $[t_0 + 2\tau, t_0 + 3\tau]$. Continuing this way leads to a solution ODE valid on $[t_0 + (k-1)\,\tau, t_0 + k\tau]$ for all $k = 4, 5, ...$

Following this procedure, a unique solution of the initial value problem can be determined.

1.2.2 Examples

To understand the method of the above steps we will present the following models.

Example 1.5 (Delayed malthusian) [6]*: For the system* (1.3) *with* $t_0 = 0$*, we assume that* $x\,(\theta) = \varphi\,(\theta)$ *for* $\theta \in [-\tau, 0]\,.$

For all $t \in [0, \tau]\,,$ *we integrate* (1.3) *on the interval* $[0, t]$ *leading to*

$$x(t) = x(0) + \int_{-\tau}^{t-\tau} rx(s)ds$$

$$x(t) = \varphi(0) + \int_{-\tau}^{t-\tau} r\varphi(s)ds \tag{1.7}$$

$$x(t) = x_1(t).$$

We repeat the same step to find of $x(t)$ *on the intervals* $[\tau, 2\tau]\,.$

$$x(t) = x(\tau) + \int_{-\tau}^{t-\tau} rx(s)ds$$

$$x(t) = x_1(\tau) + \int_{0}^{t-\tau} rx_1(s)ds$$

$$x(t) = x_2(t).$$

We repeat the same steps to obtain values of $x(t)$ *on the intervals* $[(k-1)\tau, k\tau]$ *. we get*

$$x_k(t) = x_{k-1}((k-1)\tau) + r\int_{(k-2)\tau}^{t-\tau} x_{k-1}(s)\, ds, k = 3, 4, ...$$

For an illustration of the above process, we consider the following problem

$$\begin{cases} x'(t) = rx(t-\tau), & x(t) \in \mathbb{R}, t \geq 0. \\ x(\theta) = 1, & \theta \in [-\tau, 0]. \end{cases} \tag{1.8}$$

The method of steps gives the following:

1. *For* $t \in [0, \tau]$*,*

$$x_1(t) = \varphi(0) + \int_{-\tau}^{t-\tau} r\varphi(s)\, ds = 1 + r\int_{-\tau}^{t-\tau} ds = 1 + rt.$$

2. *For* $t \in [\tau, 2\tau]$*,*

$$\begin{aligned} x_2(t) &= x_1(\tau) + \int_0^{t-\tau} rx_1(s)\, ds \\ &= 1 + r\tau + r\int_0^{t-\tau} (1+rs)ds \\ &= 1 + rt + r^2\frac{(t-\tau)^2}{2}. \end{aligned}$$

3. $x_j(t)$ *can be calculated in the same way on the intervals* $[(j-1)\tau, j\tau]$*, for all* $j = 3, ...$

The solutions obtained using the method of steps above are plotted in Figure (1.1)*. We observe that:*

- *When* $r = -1$*, the solution displays some damping oscillations.*
- *When* $r = -\frac{\pi}{2}$*, the solution is periodic.*

- *When $r = -0.5$, no oscillations.*

We can see that the magnitude of the oscillations decrease with the values of r.

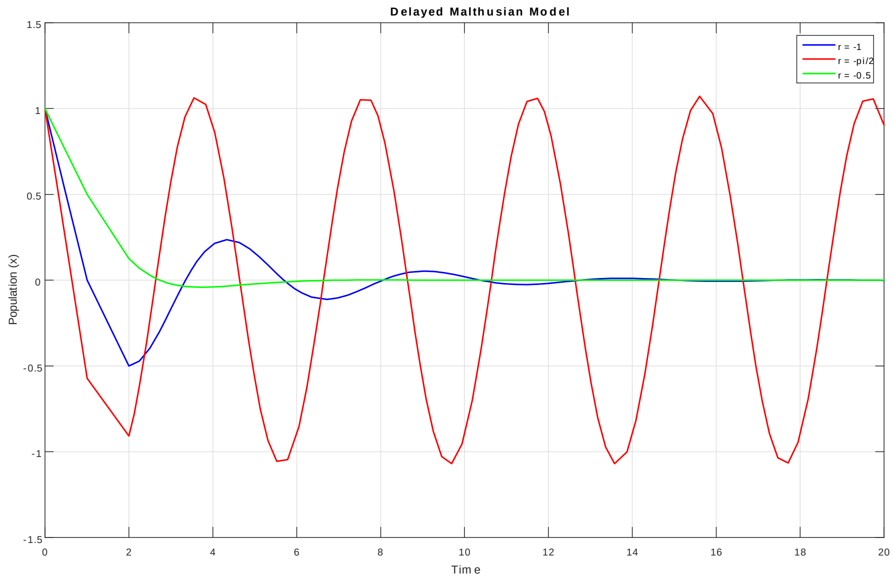

Figure 1.1: Exact solution of (1.8)for $\tau = 1$using the method of steps.

Example 1.6 (The delayed logistic model) [6] *The familiar logistic equation describing the growth of a single population is given by:*

$$\frac{dx}{dt} = rx(t)(1 - \frac{x(t)}{K}), \tag{1.9}$$

where r, K and τ are positive constants.

An alternative variant of the logistics growth model is the delayed logistics model, which adds a late factor to the population growth equation. According to this model, the rate of population growth depends on both the current and previous population sizes. Hutchinson suggested the following delayed logistic equation to account for the regulatory influence of the population from a previous time $t - \tau$.

$$\frac{dx}{dt} = rx(t)(1 - \frac{x(t-\tau)}{K}). \tag{1.10}$$

It is important to note the following points:

- *Despite its simple look, the delayed logistic model is more complex. For instance, computing the solution for $t > 0$ requires the knowledge of $x(t)$ for all $t \in [-\tau, 0]$.*
- *We cannot generally obtain explicit expressions for the solutions of this DDE.*
- *Fortunately, we can employ the method of steps to convert this equation into an ODE.*
- *However, it is not always possible to construct explicit solutions for the resulting ODE.*

Using the method of steps on the logistics model:

1. *Assume that $x(\theta) = \varphi(\theta)$ for $\theta \in [-\tau, 0]$.*
2. *For all $t \in [0, \tau]$, we have*

$$\frac{dx}{dt} = rx(t)(1 - \frac{x(t-\tau)}{K}) = rx(t)(1 - \frac{\varphi(t-\tau)}{K}). \tag{1.11}$$

3. *We denote by $x_1(t)$ the solution of the above ODE on the interval $[0, \tau]$.*
4. *We repeat this same step on the interval $[\tau, 2\tau]$ by solving*

$$\frac{dx}{dt} = rx(t)(1 - \frac{x_1(t-\tau)}{K}), \tag{1.12}$$

   leading to the solution $x_2(t)$.

It is important to note that using the method of steps for this DDE is quite demanding in terms of algebraic manipulations. One can use matlab solver dde23 to solve the logistic DDE numerically (see Figures).

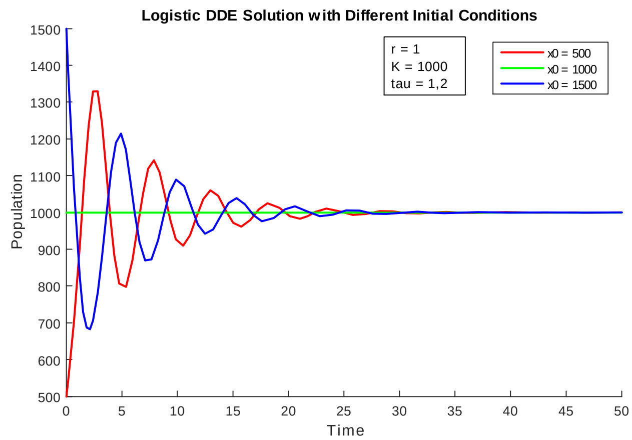

Figure 1.2: The solutions of the Delay Logistic Model (1.10) for $\tau < \frac{\pi}{2}$.

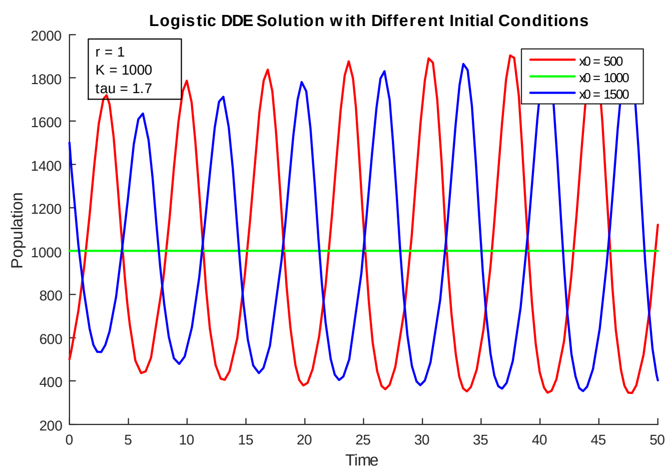

Figure 1.3: The solutions of the Delay Logistic Model (1.10) for $\tau > \frac{\pi}{2}$.

It is worth Noting that, unlike its ODE version, the delayed logistic model exhibits the following dynamics :

-The model oscillates around $x = K$ for $\tau = 1.2$.

-The model has a periodic solution for $\tau = 1.7$.

1.3 Difference between ODEs and DDEs

ODE symbolizes the ordinary differential equation, while DDE symbolizes a delayed differential equation. Both are kind of differential equations, but they differ in how systems behavior is modeled over time. Among these differences, we mention the following:

1. In ordinary differential equations (ODEs), the rate of variability change is determined by its current value and perhaps the values of other variables at the same time point. The normal differential equation is used to model dynamic systems where the system's behavior depends on the current state.

   In delayed differential equations(DDEs), the rate of variability depends on its current value and also depends on the previous value with a time delay. These equations are used to model systems in which the current situation depends on previous cases and therefore the effect of time delay enters the dynamics of the system.

   Therefore ordinary differential equations have an instantaneous effect, and generate finite dimensional system, in the other hand delayed differential equations haven't an instantaneous effect, and generate infinite dimensional system.

2. **IC'S (History function):** If we want to solve the ordinary differential equation, we need the initial value problem (IVP) of the form:

   $$x'(t) = f(t, x(t)), \quad t \geq t_0,$$

   and we need the initial conditions (IC'S) at initial time point $x(t_0) = x_0$.

   If we want to solve the delayed differential equations, we need the initial value problem (IVP) of the form

   $$x'(t) = f(t, x(t), x(t - \tau)), \quad t \geq t_0,$$

and we need the history function

$$\forall t \in [t_0 - \tau, t_0], x(t) = \varphi(t).$$

3. **Dynamical strutures:** Differential equations for delays (DDEs) already show a richer range of dynamic structures compared to ordinary differential equations (ODEs). One of the main reasons for this richness is the presence of delays, which introduce memory effects into the dynamics of the system. These memory effects can lead to a variety of complex behaviors such as oscillations and even chaotic dynamics.

   In ODEs bounded solutions of autonomous ODEs can only oscillates if there are at least two components. Also chaotic solutions only if there are at least three components.

   In DDEs, oscillatory and even chaotic behaviours can occur in the scalar case.

4. **Propagated discontinuities:** In both differential delay equations (DDEs) and Ordinary differential equations (ODEs), discontinuity can occur when there is a difference between the left and right side boundaries of the solution function derivative at a given point. This can be expressed as follows:

$$\lim_{t \to t_0^-} x_0(t)' \neq \lim_{t \to t_0^+} x_0(t)'.$$

   There exists a jump derivative discontinuity at t_0,where $x_0(t)$ is the solution function. t_0 is the point at which the discontinuity occurs. $x_0(t)'$ represents the derivative of the solution function with respect to time . This applies to both ODEs and DDEs. However, in the case of DDEs, the existence of delays leads to additional complications, and disruptions may arise due to the reliance of the system's status on previous values.

Chapter 2

Stabilty of delay differential equations

The stability analysis of the equilibrium points in the linear and nonlinear autonomous DDE is the focus of this second chapter. By using numerical simulation, a study of two systems (with 1 and 2 dimension) concludes this chapter.

Throughout this chapter we shall consider the autonomous DDE with a single delay, which is given by [4]

$$\begin{cases} x'(t) = f(x(t), x(t-\tau)) & t \geq 0, \\ x(t) = \varphi(t) & -\tau \leq t \leq 0, \end{cases} \tag{2.1}$$

where $\tau > 0$, $f \in C^1(E \times E, \mathbb{R}^n)$, $E \subseteq \mathbb{R}^n$ and $\varphi \in C\left([-\tau, 0], \mathbb{R}^n\right)$.

The solution of DDE (2.1) with initial data $\varphi(t)$ is denoted by $x(t, \varphi)$.

Our main interests is to analyze the stability equilibrium solutions of the equation (2.1) in the linear and nonlinear cases. By using numerical simulation, a study of two systems (with 1 and 2 dimension) concludes this chapter.

2.1 Stability of equilibrium points

Definition 2.1 [8] *All constant solution (steady state) $x(t,\varphi)$ of (2.1), which satisfies, $x\,(t,\varphi) = x^*$ for any $t \geq -\tau$ is called an equilibrium point.*

Therefore, to obtain all the equilibria of a DDE, we only need to solve

$$f(x^*, x^*) = 0. \tag{2.2}$$

That is if x^* is an equilibrium point of (2.1), then $x(t) = x^*$, $t \geq -\tau$ is solution of (2.1) i. e. $x'(t) = \dfrac{dx^*}{dt} = 0$ we have

$$f(x^*, x^*) = 0. \tag{2.3}$$

This equilibrium point of (2.1) are obtained by solving equation (2.3).

Notation 2.1 [2] *Let $C\left([-\tau, 0], \mathbb{R}^n\right)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We designate the norm of an element $\varphi \in C\left([-\tau, 0], \mathbb{R}^n\right)$ by*

$$\parallel \varphi \parallel_\tau = \sup_{-\tau \leq t \leq 0} \parallel \varphi(t) \parallel .$$

Definition 2.2 *:*

1. *An equilibrim point x^* of (2.1) is **stable** if for any given $\varepsilon > 0$, $\exists \delta > 0$ such that $\varphi \in C$ and*

$$\parallel \varphi - x^* \parallel_\tau < \delta \Longrightarrow \|x(t,\varphi) - x^*\| < \varepsilon, \qquad t \geq 0,$$

i.e. Nearby starting solutions stay nearby.

2. *x^* is **asymptotically stable** if it is stable and $\exists b > 0$ such that $\varphi \in C$ and*

$$\| \varphi - x^* \|_\tau < b \Longrightarrow \lim_{t\to\infty} x(t, \varphi) = x^*.$$

3. *Equilibrium point which is not stable is called **unstable**.*

The stability conditions of equilibrim points can be complex because we need the explicit solution of the given differential equations. It is more interesting and more useful to consider methods for proving stability without actually solving the differential equations. In fact we want methods which will apply to cases when we cannot (or would rather not) solve precisely.

As in the case of autonomous system of ODEs, characteristic equations are quite useful to analyse local stability. Rather than being an algebraic equation, the characteristic equation obtained from a DDE system is a transcendental equation, which means that such equation have polynomial parts and include some terms in $e^{-\lambda\tau}$.[14] [10] [2]

2.2 A linear system of delay differential equations

Definition 2.3 *A homogeneous linear autonomous system of delay differential equations with a single delay can be written as follows:*

$$x'(t) = Ax(t) + Bx(t - \tau), \tag{2.4}$$

A is a constant coefficient matrix that determines the evolution of the system without delay. B is a constant coefficient matrix representing the effect of delayed states.

To find the equilibrium points of the differential delayed equation(DDE), we need to assign a derivative of a state variable equal to zero and the solution to the values of a state variable that meets this condition.

$$x'(t) = 0 \implies Ax(t) + Bx(t - \tau) = 0.$$

Then we shall solve the equation

$$Ax^* + Bx^* = 0.$$

We see that $x^* = 0$ is an equilibrium, and it is the only equilibrium if $A + B$ is non-singular.

By the theory of dynamical system, we know that all solutions of (2.4) are (asymptotically) stable if and only if $x^* = 0$ is (asymptotically) stable.

Remember that in case of linear homogeneous system of n ordinary differential equations by fixed transactions, there are independent solutions in writing. We Know that the general solution can be expressed as arbitrary a linear combination of these solutions. but the situation is more complicated for Eq. (2.4) because, in general, (2.4) has many infinitely linear independence (already valid on $\mathbb{R}$).

2.2.1 Characteristic equation of linear delay differential equation

The characteristic equation for ODEs is a polynomial, and the number of roots, or eigenvalues, to anticipate may be found using the fundamental theorem of algebra, where as the characteristic equation for the linear DDE is transcendental, there is no theory pertaining to the number of roots (which may be countably infinite), and the study of the distinctive roots is more difficult.

In case of a scalar DDE $(n = 1)$

Let's consider the linear autonomous delayed differential equation with a single delay:

$$x'(t) = ax(t) + bx(t-\tau), \qquad x \in \mathbb{R}, \tau > 0, \tag{2.5}$$

where a and b are two real number

Same logic as what we do with ODEs, we seek exponential solutions of the form

$$x(t) = ce^{\lambda t}, \text{ where } c \neq 0 \text{ and } \lambda \in \mathbb{C}.$$

We plug it into (2.5) and get

$$\lambda e^{\lambda t} c = e^{\lambda t} a + e^{\lambda(t-\tau)} cb,$$

then we obtain

$$(\lambda - a - e^{-\lambda\tau} b)c = 0, \tag{2.6}$$

(2.6) has non-zero solution if and only if

$$\lambda - a - e^{-\lambda\tau} b = 0. \tag{2.7}$$

This equation is called transcendental characteristic equation of (2.5), and there are infinite number of solutions to this equation for complex $\lambda = \alpha + i\beta$, which all lie on curve

$$\beta = \pm\sqrt{b^2 e^{-2\tau\alpha} - (x-a)^2}.$$

In case of a system of DDEs

Let's look at the linear system of DDEs (2.4). To find the characteristic equation, we assume a solution of the form $x(t) = e^{\lambda t} v$, where $e^{\lambda t}$ represents the exponential function and $v \in \mathbb{C}^2$ is a constant vector. ($v \neq 0$). When we apply this solution to the system (2.4), we obtain:

$$\lambda e^{\lambda t} v = e^{\lambda t} A + v e^{\lambda (t-\tau)} B,$$

then we obtain

$$(\lambda I - A - e^{-\lambda \tau} B) v = 0, \tag{2.8}$$

(2.8) has non-zero solution if and only if

$$D(\lambda) := \det(\lambda I - A - e^{-\lambda \tau} B) = 0. \tag{2.9}$$

We call $D(\lambda)$ the transcendental characteristic equation of (2.4), and its roots are said to be characteristics or eigenvalues of (2.4). [10] [3] [11]

2.2.2 Stability of the linear delay differential equation

The stability theory show some important characteristics of the property of the characteristic function as in the case of a system of ordinary differential equations. Thus, in theory, we may obtain all the roots and examine each one to as certain whether an equilibrium is stable.

The following theorems show some important properties of the characteristic equation.

Theorem 2.1 [*10*] *(i) $D(\lambda)$ is an entire function.*

(ii) If λ is a characteristic root, so is $\overline{\lambda}$.

(iii) Given $\xi \in \mathbb{R}$, there are at most nitely many characteristic roots in

$$\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \xi.$$

(iv) If there are infinitely many distinct characteristic roots $\{\lambda_n\}$, *then*

$$\operatorname{Re}\lambda_n \to -\infty, n \to \infty.$$

Theorem 2.2 [10] *Suppose* $\operatorname{Re}\lambda < \mu$ *for every characteristic root* λ. *Then there exists* $k > 0$, *such that*

$$\mid x(t,\varphi) \mid < ke^{\mu t} \parallel \varphi \parallel, t \geq 0, \varphi \in C, \tag{2.10}$$

where $x(t,\varphi)$ *is the solution of* (2.4) *with initial condition* $x_0 = \varphi$. *So the equilibrium* $x = 0$ *of* (2.4) *is* ***asymptotically stable*** *if all the characteristic roots have negative real parts. On the other hand, if there exists a root with positive real part, it is* ***unstable****.*

Theorem 2.3 [9] *The following hold for the system* (2.5) .

1. *If* $a + b > 0$, *then* $x = 0$ *is unstable.*
2. *If* $a + b < 0$ *and* $b \geq a$, *then* $x = 0$ *is asymptotically stable.*
3. *If* $a + b < 0$ *and* $b < a$, *then there exists* $\tau^* > 0$ *such that* $x = 0$ *is asymptotically stable for* $0 < \tau < \tau^*$ *and unstable for* $\tau > \tau^*$.

Remark 2.1 *Transcendental equations often include an unlimited number of roots in the plane of complexity. Since we are unable to identify every root, we will need to employ a variety of techniques in order to determine the stability of equilibrium points.*

2.3 Local stability of nonlinear system of delay differential equations

[14] Consider a nonlinear autonomous system of DDEs with a single delay (2.1). That system is equivalent to:

$$\begin{cases} x_1'(t) = & f_1(x_1(t), ..., x_n(t), x_1(t-\tau)..., x_n(t-\tau)), \\ \quad . & \qquad\qquad\qquad . \\ \quad . & \qquad\qquad\qquad . \\ \quad . & \qquad\qquad\qquad . \\ x_n'(t) = & f_n(x_1(t), ..., x_n(t), x_1(t-\tau)..., x_n(t-\tau)), \end{cases}$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $f = (f_1, ..., f_n)$, $\tau > 0$, $f \in C^1(E \times E, \mathbb{R}^n)$, $E \subseteq \mathbb{R}^n$.

That system has a unique solution maximally defined for all $t > 0$ satisfying the inital condition $x(t) = \varphi(t) \in C\left([-\tau, 0], \mathbb{R}^n\right)$.

2.3.1 Linearization of nonlinear DDE

Suppose $x^* = (x_1^*, ..., x_n^*) \in \mathbb{R}^n$ is an equilibrium point of (2.1),i.e.,$f(x_1^*, ..., x_n^*) = 0$. Define $y(t) = x(t) - x^*$, where $x(t)$ be a solution of DDE, then $y(t)$ satisfies

$$\begin{aligned} y'(t) = x'(t) - \frac{dx^*}{dt} &= x'(t) - 0 \\ &= f(x(t), x(t-\tau)), \end{aligned}$$

hunce

$$y'(t) = f(x^* + y(t), x^* + y(t-\tau)). \tag{2.11}$$

To study the stability of x^*, we need to investigate the behavior of solution of (2.1) near x^*, i. e. the behavior of solution of (2.11) near $y(t) = 0$. For this purpose we expand the right hand side as the first order Taylor's approximation

$$y'(t) = f(x^*, x^*) + \left.\frac{\partial f}{\partial x\,(t)}\right|_{x=x^*} y(t) + \left.\frac{\partial f}{\partial x\,(t-\tau)}\right|_{x=x^*} y(t-\tau).$$

We obtain finaly the linearization of DDE at x^* as follows.

$$y'(t) = Ay(t) + By(t-\tau), \tag{2.12}$$

where $f(x^*, x^*) = 0$ and

$$A = \left.\left(\frac{\partial f_i}{\partial x_j(t)}\right)_{\substack{1\le i\le n\\1\le j\le n}}\right|_{(x^*,x^*)} = \left.\left(\frac{\partial f}{\partial x(t)}\right)_{\substack{1\le i\le n\\1\le j\le n}}\right|_{(x^*,x^*)},$$

and

$$B = \left.\left(\frac{\partial f_i}{\partial x_j(t-\tau)}\right)_{\substack{1\le i\le n\\1\le j\le n}}\right|_{(x^*,x^*)} = \left.\left(\frac{\partial f}{\partial x(t-\tau)}\right)_{\substack{1\le i\le n\\1\le j\le n}}\right|_{(x^*,x^*)},$$

are $n \times n$ matrices evaluated at the equilibrium point.

Then the linear stability of $x^*$of system (2.1) is comparable to the zero answer found in system (2.12) is stable similar to the ODE case. We seek the solution $y(t) = e^{\lambda t}v$, $v \neq 0$, It simple to confirm that $y(t) = e^{\lambda t}v$, $v \neq 0$, is a solution of (2.12) if and only λ a solution of the characteristic equation.

$$D(\lambda) := \det(\lambda I - A - e^{-\lambda\tau}B) = 0. \tag{2.13}$$

Equation (2.13) has an infinite number of roots $\lambda \in \mathbb{C}$ which determine the stability of steady state solution x^*. The following result reveals the relation of system (2.1) and its linearized system (2.12). [10] [14] [8] [3]

Theorem 2.4 (Local stability of nonliner autonomous DDEs) [10]*: Let $D(\lambda)$ be the characteristic equation corresponding to (2.12), then $x^*$is* ***locally asymptotically stable*** *if every root of $D(\lambda)$ has negative real part. In fact, there exit $\delta > 0$, $k > 0$ such that*

$$\| \varphi - x^* \| < \delta \Rightarrow \| x_t(\varphi) - x^* \| \leq k \| \varphi - x^* \| e^{-\mu t}, t \geq 0,$$

where

$$-\mu := \sup_{D(\lambda)=0} \operatorname{Re} \lambda < 0.$$

On the other hand, x^* is unstable if one of the roots of $D(\lambda)$ has positive real part.

Definition 2.4 (Classification of equilibrium points) *: Let x^* an equilibrium point of system (2.1) and $D(\lambda)$ be the characteristic equation corresponding to (2.12).*

1. *The equilibrium point x^* is called a* ***sink*** *if all roots λ of $D(\lambda)$ satisfy*

$$\operatorname{Re}(\lambda) < 0.$$

2. *The equilibrium point x^* is called a* ***source*** *if all roots λ of $D(\lambda)$ satisfy*

$$\operatorname{Re}(\lambda) > 0.$$

3. *The equilibrium point x^* is called a* ***saddle*** *if at least one root of $D(\lambda)$ has negative real part and at least one has positive real part and no one has a zero real.*

Lemma 2.3.1 *:*

1. *All* ***sinks*** *are* ***asymptotically stable*** *equilibrium points.*

2. *All* ***sources*** *and* ***saddles*** *are* ***unstable*** *equilibrium points.*

Example 2.1 [15] *Consider the delay differential equation with $\tau = 1$ given by*

$$x'(t) = -x(t) - x^2(t-1).$$

1. *To identify the equilibrium points : We have*

$$f(x(t), x(t-\tau)) = -x(t) - x^2(t-1),$$

then equilibrium points are the solution of

$$f(x^*, x^*) = -x^* - {x^*}^2 = -x^*(1 + x^*) = 0.$$

So the system has two equilibrium points $x_1^* = 0$ and $x_2^* = -1$.

2. *Equilibrium points classification*

For $x_1^* = 0$, we have, $\left(\frac{\partial f}{\partial x(t)}\right)\Big|_{(0,0)} = -1$, and $\left(\frac{\partial f}{\partial x(t-1)}\right)\Big|_{(0,0)} = 0 \Rightarrow a = -1$ and $b = 0$. Then the linearized system is

$$y'(t) = -y(t), \tag{2.14}$$

and the characteristic equation of (2.14) is given by

$$\lambda + 1 = 0 \implies \lambda = -1.$$

Because $\operatorname{Re}(\lambda) < 0$, the equilibrium points $x_1^* = 0$ is a ***sink****, which is* ***asymptotically stable.***

For $x_2^* = -1$, *we have*

$$\left(\frac{\partial f}{\partial x(t)}\right)\Big|_{(-1,-1)} = -1, \ \ and \ \left(\frac{\partial f}{\partial x(t-1)}\right)\Big|_{(-1,-1)} = 2 \Rightarrow a = -1 \ and \ b = 2.$$

Then the linearized system is

$$y'(t) = -y(t) - 2y(t-1), \tag{2.15}$$

and the characteristic equation of (2.15) *is given by*

$$\lambda + 1 - 2e^{-\lambda} = 0.$$

There exist roots $\lambda_1 = 0,3748 > 0$ *and* $\lambda_2 = -0,86 + 4,74i$. *Because* $\operatorname{Re}(\lambda_1) > 0$, *the equilibrium points* $x_2^* = -1$ *is a* ***saddle****, which is* ***unstable.***

2.3.2 Delay effects on stability

[14] Among the methods used to study the stability is the geometric approach in the case where the DDE system (2.1) has a characteristic equation at the equilibrium point x^* given by

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \tag{2.16}$$

where P and Q are polynomial in λ.

As we know that the equilibrium x^* is asymptotically stable if all the roots λ of the equation (2.16) satisfy $\operatorname{Re}(\lambda) < 0$, and is unstable if there exists a root λ such that $\operatorname{Re}(\lambda) > 0$. Therefore the change in stability can occur only if a root λ of equation (2.16) crosses the imaginary axis.

Suppose that in the case $\tau = 0$, the equilibrium point x^* is asymptotically stable.

Let now $\lambda = is$ with $(s \geq 0)$ be a root of (2.16), then we get

$$\frac{P(is)}{Q(is)} = -e^{-is\tau}. \tag{2.17}$$

The right side of (2.17) traces aut aunit circle in the complex plane on the other hand, the left-hand side of (2.17) also defines a curve called ratio curve. The change in stability can accur only if the ratio curve intersect the unit circle, that is if there exist positive numbers s^* and τ^* for which the equation (2.17) holds as $s^*\tau^*$ increased from 0 to 2π.

We find s^* such that

$$\left|\frac{P(is^*)}{R(is^*)}\right| = 1,$$

and the critical value τ^* is defined by

$$\tau^* = \frac{-1}{is^*} \log\left[\frac{-P(is^*)}{R(is^*)}\right].$$

Then for $0 < \tau < \tau^*$, the equilibrium point is asymptotically stable.

2.3.3 Applications on the stability

Local stability of DDEs is more challenging than for ordinary DEs, due to the infinite dimensionality of the system. In this paragraph, we consider these two examples.

Example 2.2 [7] *Consider the linear delay-differential equation in dimensionless form.*

$$\frac{dx}{dt} = hx(t-1), \qquad h \in \mathbb{R}. \tag{2.18}$$

The system has only one equilibrium point $x^* = 0$. If we consider the exponential solution $x = ce^{\lambda t}$ so that

$$\frac{dx}{dt} = c\lambda e^{\lambda t}, \tag{2.19}$$

also

$$hx(t-1) = hce^{\lambda(t-1)} = hce^{\lambda t}e^{-\lambda}, \tag{2.20}$$

together they give characteristic equation as

$$\lambda - he^{-\lambda} = 0. \tag{2.21}$$

Since the characteristic equation for the linear DDE is transcendental, there is no theory about the number of roots, so we consider real and complex solutions separately.

1. *Suppose that λ is real, we can plot $z_1 = \lambda$ and $z_2 = he^{-\lambda}$ and look for intersections.*

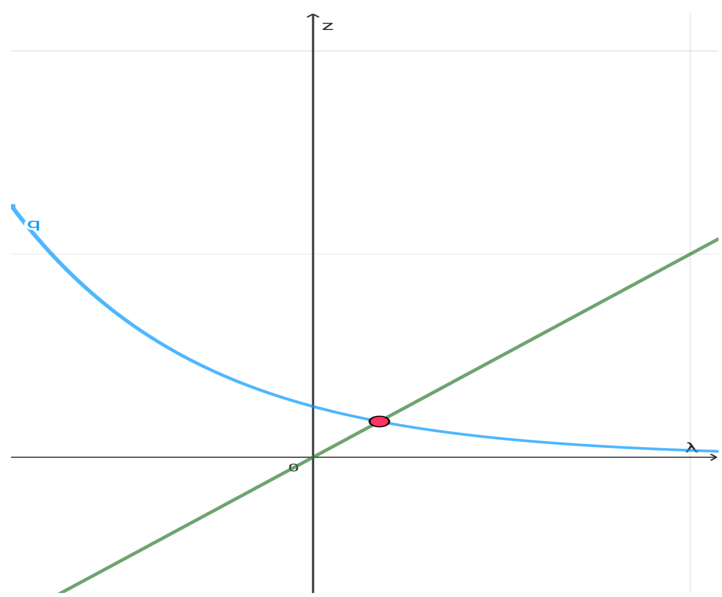

Figure 2.1: Intersection of $z_1 = \lambda$ (in green) and $z_2 = he^{-\lambda}$(in blue), for real λ and $h > 0$.

For $h > 0$,there is a single intresection at a positive λ .Thus, the solution $x = ce^{\lambda t}$ increases exponentially to infinity as $t \to \infty$, and the equilibrium $x^* = 0$ is unstable.

For $h < 0$, there may be 0, 1, or 2 intersections. There is a single intersection *when the curves are tangent. At this point of tangency the curves have the same slope, and since the line has slope of* 1*, this means that at the point of tangency*

$$\frac{d}{d\lambda}he^{-\lambda} = 1,$$

or

$$-he^{-\lambda} = 1,$$

so that

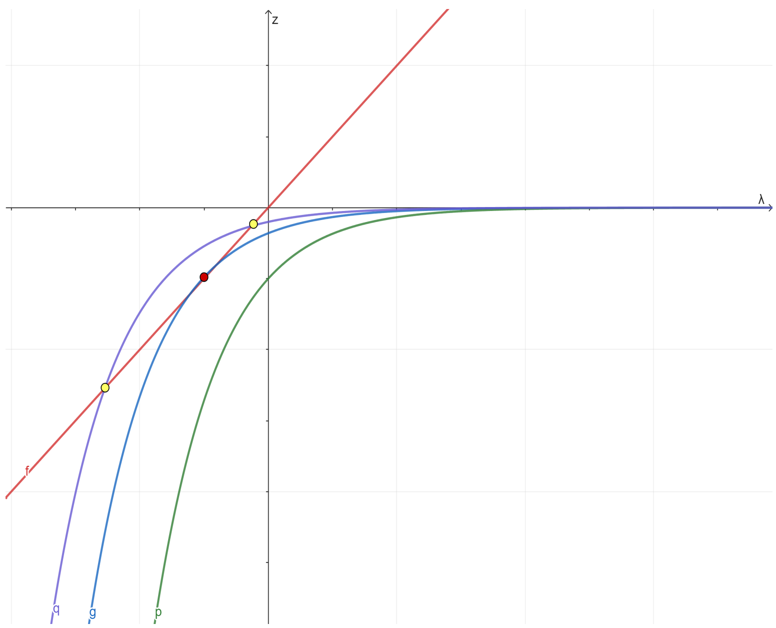

Figure 2.2: For λ real.and $h < 0$ Intersections of $z_1 = \lambda$ (in red) and $z_2 = he^{-\lambda}$, for $h_c < h < 0$.(in violet), for $h = h_c$ (in blue) and for $h < h_c$ (in green).

$$h = -e^{\lambda}.$$

Since this point is on the curve $z = he^{-\lambda}$, if we replace h, then we get $z = -1$, and since the intersection is on the line $z = \lambda$, then $\lambda = -1$ also.

Therefore, the tangency occurs (and there is a single intersection) when

$$h_c = -e^{-1}.$$

- *For* $h_c < h < 0$, *there are two real negative eigenvalues.*
- *For* $h = h_c = -e^{-1}$, *there is a single negative eigenvalue.*
- *For* $h < h_c$, *there are no real eigenvalues.*

By superposition, the solution to the linear DDE is the sum of exponential solutions. So, if $h \in [h_c, 0]$ *the real eigenvalues are negative and so the associated exponential solutions decay to* 0 *over time. If* $h < h_c$ *there are no exponentially decaying or growing components to the solution.*

2. *Suppose that* λ *is comlex. then substiute* $\lambda = \lambda_r + \iota\lambda_i$ *into characteristic equation (Eq. 2.21)). we have*

$$\begin{aligned}\lambda_r + \iota\lambda_i &= he^{-\lambda_r - \iota\lambda_i} \\ &= he^{-\lambda_r} e^{-\iota\lambda_i} \\ &= he^{-\lambda_r}[\cos(-\lambda_i) + \iota\sin(-\lambda_i)] \\ &= he^{-\lambda_r}\cos(\lambda_i) - \iota he^{-\lambda_r}\sin(\lambda_i).\end{aligned}$$

This implies that

$$\begin{cases} \lambda_r = he^{-\lambda_r}\cos(\lambda_i), \\ \lambda_i = -he^{-\lambda_r}\sin(\lambda_i). \end{cases} \tag{2.22}$$

Therefore

$$\frac{\lambda_r}{\lambda_i} = -\cot(\lambda_i), \tag{2.23}$$

using (Eq(2.22)) and (Eq(2.23)) we get parametric equations for h *with* λ_i *as a parameter:*

$$h = -\frac{\lambda_i}{e^{\lambda_i \cot(\lambda_i)}\sin\lambda_i}. \tag{2.24}$$

Due to the periodicity of the trigonometric functions, many curves are traced out as λ_i *is varied from* $-\infty$ *to* ∞, *producing an infinite number of curves.*

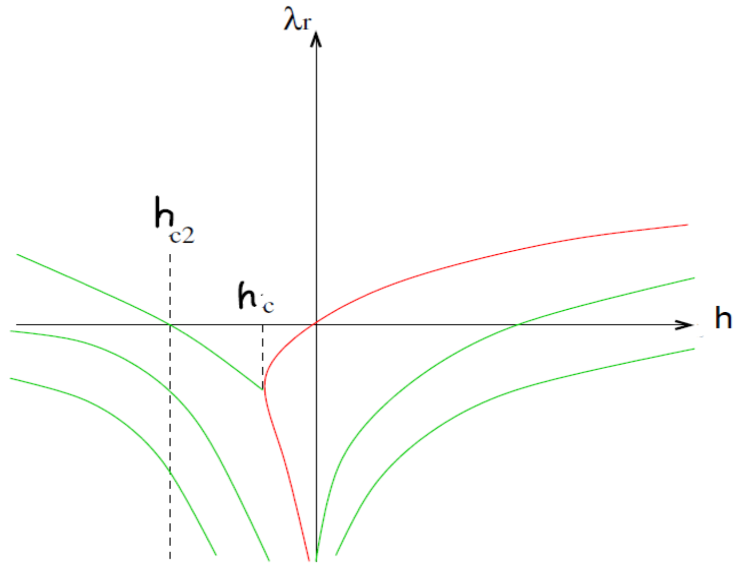

Figure 2.3: For complex λ. Infinite number of solution curves (in green). The solution curve for real (in red).

The general solution is $x(t;h) = \sum c_n e^{\lambda_n t}$*,where summation is over all values of* λ_n *for the given parameter* h *,and* $x^* = 0$ *is a* ***stable equilibrium point*** *for values of* h *in which all eigenvalues have negative real parts. That is, for all values of* h *between the dashed line and* $h = 0$*,* $h \in (h_{c_2}, 0)$ *in the figure, where* h_{c_2} *is the value of* h *in which* $\lambda_r = 0$*, and* $\lambda_i \neq 0$ *(the eigenvalue is complex).*

From the first equation in. (2.22) *if* $\lambda_r = 0$*, then*

$$0 = h\cos(\lambda_i), \tag{2.25}$$

or

$$\lambda_i = \pm\frac{\pi}{2} + k\pi, k \in \mathbb{Z}, \tag{2.26}$$

and using the second equation in (2.22) ,

$$\lambda_i = -h\sin(\lambda_i), \tag{2.27}$$

or

$$\lambda_i = \pm h, \tag{2.28}$$

and thus

$$h = \pm \lambda_i. \tag{2.29}$$

*Equations (2.26) and (2.29) give the values of h where each of an infinite number of complex eigenvalues cross through $\lambda_r = 0$. We don't care about positive h values, since we know that in this case the real eigenvalue is positive, and hence $x^* = 0$ **is unstable.** For the negative values of h, the first crossing is at $\lambda_i = \frac{\pi}{2}$ and from the equation. (2.29), $h_{c_2} = -\frac{\pi}{2}$. Therefore, $x^* = 0$ **is stable** for $h \in (-\frac{\pi}{2}, 0)$.*

Example 2.3 [15] *Consider the two dimensional system of delayed differential equations*

$$\begin{cases} x'(t) = -x^2(t) - y(t-\tau), \\ y'(t) = x(t) - 3y(t) + 2x(t-\tau)^2. \end{cases} \tag{2.30}$$

1. *To identify the equilibrium points : We have to solve the equation*

$$\begin{cases} {x^*}^2 + y^* = 0, \\ x^* - 3y^* + 2{x^*}^2 = 0. \end{cases}$$

   So, we obtain

$$\begin{cases} x_1^* = 0 \text{ and } y_1^* = 0, \\ \text{or} \\ x_2^* = -\dfrac{1}{5} \text{ and } y_2^* = -\dfrac{1}{25}. \end{cases}$$

   Therefore the system (2.30) has twe equilibrium points $E_1 = (0,0)$ and $E_2 \left(-\dfrac{1}{5}, -\dfrac{1}{25}\right)$.

2. *To analyze stability of equilibrium point $E_1 = (0,0)$ of system (2.30). we'll use the geometric method.*

The jacobian matrices are given by

$$A = \begin{bmatrix} -2x(t) & 0 \\ 1 & -3 \end{bmatrix}\Bigg|_{(x^*,y^*)} = \begin{bmatrix} -2x^* & 0 \\ 1 & -3 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 & -1 \\ 4x(t-\tau) & 0 \end{bmatrix}\Bigg|_{(x^*,y^*)} = \begin{bmatrix} 0 & -1 \\ 4x^* & 0 \end{bmatrix}.$$

For the stability of $E_1 = (0,0)$, *we have*

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}, \quad \textit{and} \quad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

then we can get the linearized equation

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + B \begin{bmatrix} x(t-\tau) \\ y(t-\tau) \end{bmatrix}. \tag{2.31}$$

When $\tau = 0$, *the system* (2.31) *becomes*

$$X'(t) = (A+B)X(t), \tag{2.32}$$

where $X = (x,y)$ *and the matrix* $A+B$ *are given by*

$$A+B = \begin{bmatrix} 0 & -1 \\ 1 & -3 \end{bmatrix}.$$

So the characteristic polynome is

$$\det(\lambda I - A - B) = 0 \implies \lambda^2 + 3\lambda + 1 = 0,$$

which has two real negative roots as follows

$$\lambda_{1,2} = \frac{-3 \pm \sqrt{5}}{2} < 0.$$

Therefore the system (2.32) *is asymptotically stable.*

In case of positive delay, i.e. $\tau > 0$*, The chracteristic equation for the linearized equation around the point* E_1 *is given by*

$$\det(\lambda I - A - Be^{-\lambda\tau}) = 0 \implies \begin{vmatrix} \lambda & e^{-\lambda\tau} \\ -1 & \lambda + 3 \end{vmatrix} = 0 \implies \lambda^2 + 3\lambda + e^{-\lambda\tau} = 0. \tag{2.33}$$

Let $\lambda = is$, $(s > 0)$ *be a purely root of* (2.33)*, and by substituting we get*

$$(is)^2 + 3is + e^{-is\tau} = 0,$$

i. e.

$$-s^2 + 3is = -e^{-is\tau}, \tag{2.34}$$

the right side of equation (2.34) *is unit circle and the left side is a ratio carve. The ratio curve intersects the unit circle if*

$$\begin{aligned} &\left|-s^2 + 3is\right| = 1 \Rightarrow s^4 + 9s^2 = 1 \Rightarrow (s^2)^2 + 9s^2 - 1 = 0, \\ &\qquad \implies s^2 = \frac{-9 \pm \sqrt{85}}{2} \implies s^2 \simeq 0.10977 \implies s^* \simeq 0.331319. \end{aligned}$$

So the critical value τ^* *is give when we substitute the value of* s^* *in the equation* (2.34) :

$$-0.10977 + 3i(0,331319) = -\cos(0.331319\tau^*) + i\sin(0.331319\tau^*).$$

So

$$\sin(0.331319\tau^*) = 3 \times 0.331319 \simeq 1,$$

which implay

$$0.331319\tau^* = \frac{\pi}{2} \Rightarrow \tau^* = \frac{\pi}{2 \times 0.331319}\ .$$

Therefore the equilibrium point E_1 is asymptotically stable if $0 < \tau < \tau^*$. (see Figure).

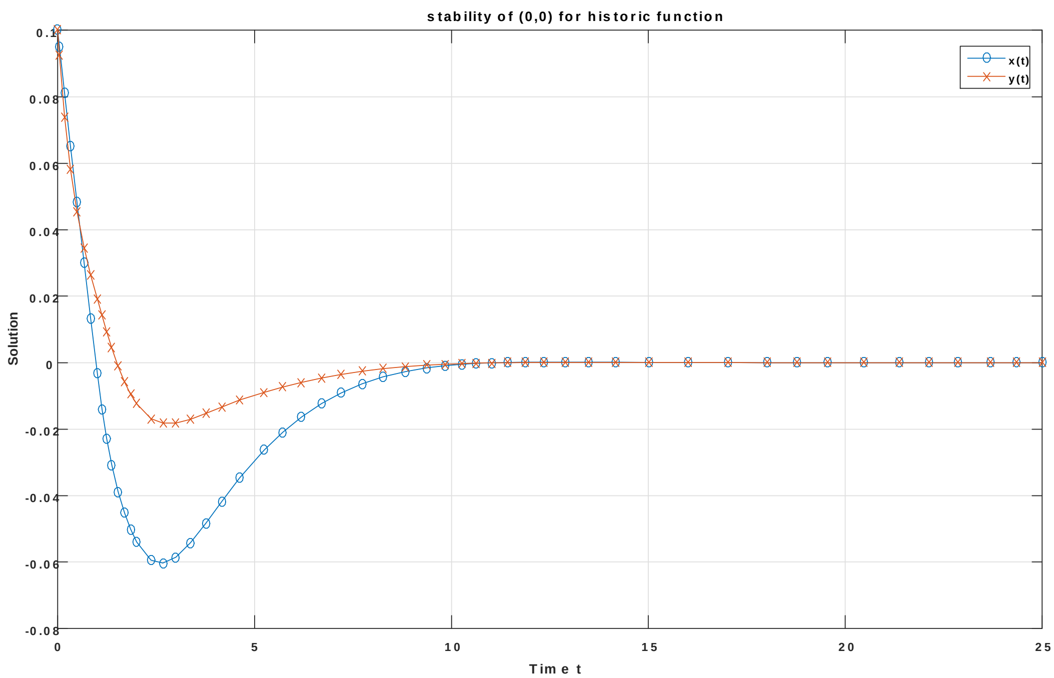

Figure 2.4: Stability of $E_1 = (0, 0)$ where the historic function $(0.1, 0.1)$ and $\tau = 1$.

Conclusion

Delay differential equations (DDEs) provide more realistic models in various fields as biology, chemistry, physics, and economics. This work presents the existence and uniqueness theorem, which ensures a unique solution under certain conditions, using the history function rather than initial conditions as in ordinary differential equations (ODEs). It is important to note that using the step method for solving DDEs analytically is quite demanding in terms of algebraic manipulations. To solve these DDEs numerically, we can use MATLAB's dde23 solver.

The stability analysis of linear and non-linear delay differential equations with a single delay near equilibrium points can be done by the study of the roots of a transcendental characteristic equation, whereas there is no general theorem on the number of these roots which could be infinite. Therefore we present a geometric approach as a method to study the stability and to determine the effect of delay. We conclude our study with numerical simulations of one- and two-dimensional systems.

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Annexe A: Matlab codes for chapter 1

The following programms solve the solution of the delayed logistic model using dde23.

Figure 1.3

```
% Define the parameters

r = 1; % Growth rate

K = 1000; % Carrying capacity

tau = 1.2; % Time delay

% Define the DDE function

dde_eq = @(t, x, x1) r * x * (1 - x1(1, end)/K);

% Define the time interval

tspan = [0 50];

% Define different initial conditions

x0_values = [500, 1000, 1500];

hold on; % Keep the plot active for multiple curves

% Loop through different initial conditions

for i = 1:numel(x0_values)

% Define the history function (initial condition)
```

```
x0 = @(t) x0_values(i);

% Solve the DDE for the current initial condition

sol = dde23(dde_eq, tau, x0, tspan);

% Plot the solution

plot(sol.x, sol.y, 'LineWidth', 1.5); % Adjust line width for clarity

end

xlabel('Time')

ylabel('Population')

title('Logistic DDE Solution with Different Initial Conditions')

legend('x0 = 500', 'x0 = 1000', 'x0 = 1500') % Add legend for clarity
```

Figure 1.4:

```
% Define the parameters

r = 1; % Growth rate

K = 1000; % Carrying capacity

tau = 1.7; % Time delay

% Define the DDE function

dde_eq = @(t, x, x1) r * x * (1 - x1(1, end)/K);

% Define the time interval

tspan = [0 50];

% Define different initial conditions

x0_values = [500, 1000, 1500];

hold on; % Keep the plot active for multiple curves

% Loop through different initial conditions

for i = 1:numel(x0_values)
```

```
% Define the history function (initial condition)
x0 = @(t) x0_values(i);
% Solve the DDE for the current initial condition
sol = dde23(dde_eq, tau, x0, tspan);
% Plot the solution
plot(sol.x, sol.y, 'LineWidth', 1.5); % Adjust line width for clarity
end
xlabel('Time')
ylabel('Population')
title('Logistic DDE Solution with Different Initial Conditions')
legend('x0 = 500', 'x0 = 1000', 'x0 = 1500') % Add legend for clarity
```

Annexe B: Abbreviations and Notations

DDEs	Delay differential equations.
ODEs	ordinary differential equations.
$x(t)$	The state of the system at time t.
$x(t-\tau)$	The state of the system at a previous time $t-\tau$.
τ	The delay.
$\varphi(t)$	The history function.
$f_x(t,x,y)$	the jacobian matrix with respect to x.
$f_y(t,x,y)$	the jacobian matrix with respect to y.
v	constant vector.
$D(\lambda)$	the transcendental characteristic equation.
$\mathrm{Re}(\lambda)$	the real part of a complex number λ.

Abstract

The study of delay differential equations (DDEs) aims to understand the behavior of systems where the current state depends not only on present conditions but also on past states. The main objective of this work is to discuss the existence and uniqueness of solutions and give an analytic method to solve DDEs. Furthermore, we aim to study the stability of both linear and nonlinear equations to determine the conditions under which the system remains stable. The analysis will include examining equilibrium points and characteristic equations.

Keywords: **delay differential equations, historic function , method of steps, existence and uniqueness of solutions, stability theory**.

Résumé

L'étude des équations différentielles à retard (DDEs) vise à comprendre le comportement des systèmes où l'état actuel dépend non seulement des conditions présentes mais aussi des états passés. L'objectif principal de ce travail est de discuter l'existence et l'unicité des solutions et de donner une méthode analytique pour résoudre les DDEs. De plus, nous présentons l'étude de la stabilité des équations linéaires et non linéaires pour déterminer les conditions dans lesquelles le système reste stable. L'analyse comprendra l'examen des points d'équilibre et des équations caractéristiques.

Mots-clés : **équations différentielles à retard, fonction historique, méthode des étapes, existence et unicité des solutions, théorie de stabilité.**

الملخص

تهتم دراسة المعادلات التفاضلية ذات تأخير إلى فهم سلوك الأنظمة حيث لا تعتمد الحالة الحالية على الظروف الحالية فقط ولكن أيضًا على الحالة في الزمن الماضي. الهدف الرئيسي من هذا العمل هو مناقشة وجود وحدانية الحلول وإعطاء طريقة تحليلية لحل المعادلات التفاضلية ذات تأخير. علاوة على ذلك، نتطرق إلى دراسة استقرار كل من المعادلات الخطية وغير الخطية لتحديد الظروف التي يظل فيها النظام مستقرًا. سيتضمن التحليل فحص نقاط التوازن والمعادلات المميزة.

الكلمات المفتاحية: **المعادلات التفاضلية ذات تأخير، دالة الماضي، طريقة الخطوات، وجود ووحدانية الحلول، نظرية الاستقرار.**